ATLANTA PUBLIC SCHOOLS
Mathematics & Science Initiative
Making A Difference

Atlanta Public Schools

Teacher's Curriculum Supplement

Common Core Georgia Performance Standards Mathematics III

Unit 5: Circles and Parabolas

GE Foundation

This document has been made possible by funding from the GE Foundation Developing Futures grant, in partnership with Atlanta Public Schools. It is derived from the Georgia Department of Education Math III Framework and includes contributions from Georgia teachers. It is intended to serve as a companion to the GA DOE Math III Framework Teacher Edition. Permission to copy for educational purposes is granted and no portion may be reproduced for sale or profit.

Preface

We are pleased to provide this supplement to the Georgia Department of Education's Mathematics III Framework. It has been written in the hope that it will assist teachers in the planning and delivery of the new curriculum, particularly in these first years of implementation. This document should be used with the following considerations.

- The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics teachers should work the tasks, read the teacher notes provided in the Georgia Department of Education's Mathematics III Framework Teacher Edition, and *then* examine the lessons provided here.

- This guide provides day-by-day lesson plans. While a detailed scope and sequence and established lessons may help in the implementation of a new and more rigorous curriculum, it is hoped that teachers will assess their students informally on an on-going basis and use the results of these assessments to determine (or modify) what happens in the classroom from one day to the next. Planning based on student need is much more effective than following a pre-determined timeline.

- It is important to remember that the Georgia Performance Standards provide a balance of concepts, skills, and problem solving. Although this document is primarily based on the tasks of the Framework, we have attempted to help teachers achieve this all important balance by embedding necessary skills in the lessons and including skills in specific or suggested homework assignments. The teachers and writers who developed these lessons, however, are not in your classrooms. It is incumbent upon the classroom teacher to assess the skill level of students on every topic addressed in the standards and provide the opportunities needed to master those skills.

- In most of the lesson templates, the sections labeled *Differentiated support/enrichment* have been left blank. This is a result of several factors, the most significant of which was time. It is hoped that as teachers use these lessons, they will contribute their own ideas, not only in the areas of differentiation and enrichment, but in other areas as well. Materials and resources abound that can be used to contribute to the teaching of the standards.

On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

We hope that teachers will incorporate these strategies in each and every lesson.

It is hoped that you find this document useful as we strive to raise the mathematics achievement of all students in our district and state. Comments, questions, and suggestions for inclusions related to this document may be emailed to Dr. Dottie Whitlow, Executive Director, Mathematics and Science Department, Atlanta Public Schools, dwhitlow@atlantapublicschools.us.

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Explanation of the Terms and Categories Used in the Lesson Template

Task: This section gives the suggested number of days needed to teach the concepts addressed in a task, the task name, and the problem numbers of the task as listed in the Georgia Department of Education's Mathematics III Framework Teacher Edition (GaDOE TE).

In some cases new tasks or activities have been developed. These activities have been named by the writers.

Standard(s): Key standards addressed in the lesson are listed in this section. Standards listed first, in regular type, are from the Common Core State Standards for Mathematics. Standards in bold type are the corresponding standards from Mathematics III of the Georgia Performance Standards.

New Vocabulary: Vocabulary is listed here the *first* time it is used. It is strongly recommended that teachers, particularly those teaching Math Support, use interactive word walls. Vocabulary listed in this section should be included on the word walls and previewed in Math Support.

Mathematical concepts/skills: Major concepts addressed in the lesson are listed in this section whether they are CCGPS Math III concepts or were addressed in earlier grades or courses.

Prior knowledge: Prior knowledge includes only those topics studied in previous grades or courses. It does not include CCGPS Math III content taught in previous lessons.

Essential Question(s): Essential questions may be daily and/or unit questions.

Suggested materials: This is an attempt to list all materials that will be needed for the lesson, including consumable items, such as graph paper; and tools, such as graphing calculators and compasses. This list does not include those items that should always be present in a standards-based mathematics classroom such as markers, chart paper, and rulers.

Warm-up: A suggested warm-up is included with every lesson. Warm-ups should be brief and should focus student thinking on the concepts that are to be addressed in the lesson.

Opening: Openings should set the stage for the mathematics to be done during the work time. The amount of class time used for an opening will vary from day-to-day but this should not be the longest part of the lesson.

Worktime: The problem numbers have been listed and the work that students are to do during the worktime has been described. A student version of the day's activity follows the lesson template in every case. In order to address all of the standards in CCGPS Math III, some of the problems in some of the original GaDOE tasks have been omitted and less time consuming activities have been substituted for those problems. In many instances, in the student versions of the tasks, the writing of the original tasks has been simplified. In order to preserve all vocabulary, content, and meaning it is important that teachers work the original tasks as well as the student versions included here.

Teachers are expected to both facilitate and provide some direct instruction, when necessary, during the work time. Suggestions related to student misconceptions, difficult concepts, and deeper meaning have been included in this section.

Questioning is extremely important in every part of a standards-based lesson. We included suggestions for questions in some cases but did not focus on providing good questions as extensively as we would have liked. Developing good questions related to a specific lesson should be a focus of collaborative planning time.

Closing: The closing may be the most important part of the lesson. This is where the mathematics is formalized. Even when a lesson must be continued to the next day, teachers should stop, leaving enough time to “close”, summarizing and formalizing what students have done to that point. As much as possible students should assist in presenting the mathematics learned in the lesson. The teacher notes are all important in determining what mathematics should be included in the closing.

Homework: In some cases, homework suggestions are provided. Teachers should use their resources, including the textbook, to assign homework that addresses the needs of their students.

Homework should be focused on the skills and concepts presented in class, relatively short (30 to 45 minutes), and include a balance of skills and thought-provoking problems.

Differentiated support/enrichment: On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution
- flexible grouping of students
- multiple representations of mathematical concepts
- writing in mathematics
- monitoring of progress through on-going informal and formative assessments
- and analysis of student work.

Check for understanding: A check for understanding is a short, focused assessment-a ticket out the door, for example. There are many good resources for these items on-line at www.georgiastandards.org, along with other GaDOE materials related to the standards.

Resources/materials for Math Support: Again, in some cases, we have provided materials and/or suggestions for Math Support. This section should be personalized to your students, class, and/or school, based on your resources.

Table of Contents
Unit 5: Circles and Parabolas

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Unit 5 Timeline

Task 1: The Algebra of Circles	3 days
Mini-tasks: Applying the Algebra of Circles to Real Life Situations	1 day
Task 2: Parabolas	2 days

Task Notes

The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Common Core Georgia Performance Standards in Mathematics, teachers should work the Student Tasks, read any corresponding teacher notes, and *then* examine the lessons provided here.

The tasks provided in this Supplement are based on the content (related to circles and parabolas) of Unit 5 of the Georgia Department of Education's Mathematics III Framework. We suggest, as always, that teachers use this Supplement along with the GaDOE Teacher Edition which can be found on *Learning Village* on-line at www.georgiastandards.org.

Task 1: *The Algebra of Circles*
The concepts and skills addressed in this task include:

- deriving the equation of a circle of given center and radius using the Pythagorean Theorem
- converting from general to standard form of an equation of a circle by completing the square
- determining the center and radius of a circle given its equation
- graphing a circle in standard form by hand and by using appropriate technology
- solving systems of equations involving a circle and a line algebraically, graphically, and using appropriate technology
- writing systems of equations involving circles and lines that satisfy given conditions (for example, a circle and a line that have no points in common).
- deriving the equation of a line tangent to a circle at a given point
- writing the equation of a circle tangent to a given line
- solving systems of equations involving two circles

Items in this task have been used as written or adapted from the task entitled *Circles and Radio Stations Learning Task* from Unit 5 the GaDOE Mathematics III Framework.

Mini-tasks 1 and 2: Radio Stations and Crop Circles
These mini-tasks provide excellent opportunities for students to apply what they have learned about circles to real situations. They may be used as deemed appropriate. We strongly suggest that teachers require students to complete the *Radio Station* mini-task either individually or in groups.

Task 2: *Parabolas*
The concepts and skills addressed in this task include:

- identifying the vertex, focus, directrix, and axis of symmetry of a parabola, given its equation
- graphing a parabola given its equation
- writing the equation of a parabola given characteristics of its graph
- converting from general to vertex form of an equation of a parabola by completing the square

Items in this task have been used as written or adapted from the task entitled *Parabolas Learning Task* from Unit 5 the GaDOE Mathematics III Framework.

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Task 1:
The Algebra of Circles

CCGPS Mathematics III
Task 1: The Algebra of *Circles* **Day 1/3**
(GaDOE Task: *Circles and Radio Stations Learning Task, Items 1-5)*

CCSS Standard(s):
Algebra
Reasoning with Equations and Inequalities A-REI
Represent and solve equations and inequalities graphically
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Geometry
Expressing Geometric Properties with Equations G-GPE
Translate between the geometric description and the equation for a conic section
1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically
4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,* $\sqrt{3}$*) lies on the circle centered at the origin and containing the point (0, 2).*

GPS Standard(s):
MM3G1. Students will investigate the relationships between lines and circles.
a. Find equations of circles.
b. Graph a circle given an equation in general form.

New vocabulary: general form of an equation of a circle, standard form of an equation of a circle, completing the square

Mathematical concepts/skills:
- deriving the equation of a circle of given center and radius using the Pythagorean Theorem
- converting from general to standard form of an equation of a circle by completing the square
- determining the center and radius of a circle given its equation
- graphing a circle in standard form by hand and by using appropriate technology

Prior knowledge:
- definition of a circle
- Pythagorean Theorem
- transformations of functions, including horizontal and vertical shifts

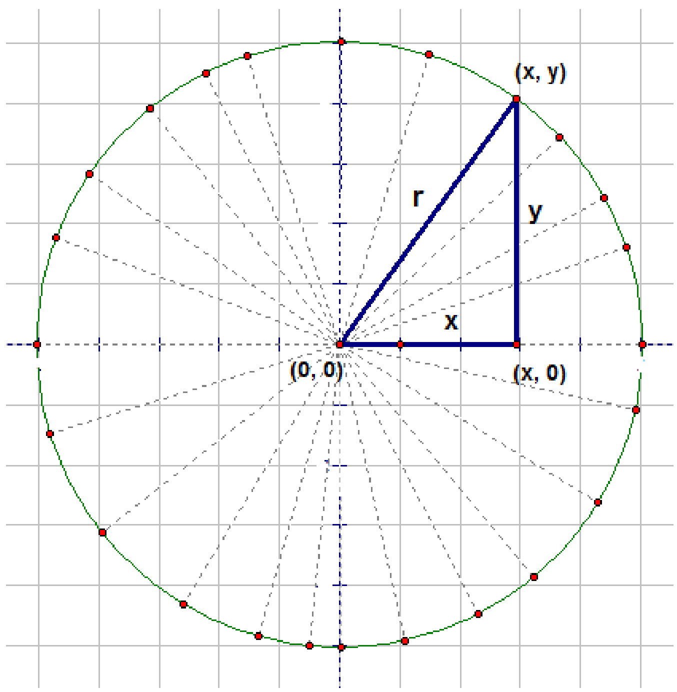
Essential question(s): How can I determine the center and radius of a circle given its equation? How can I write an equation of a circle given certain characteristics of the circle?

Suggested materials:

- graph paper
- graphing calculators

Warm-up: Post the following:

The diagram below shows a circle with center (0, 0) and radius r. The point (x, y) lies on the circle. Write an equation that expresses the relationship between x, y, and r. How does this equation relate to the circle?



Opening: Discuss student responses to the *Warm-up*. Hopefully some students will use the diagram and the Pythagorean Theorem to express the relationship between *x*, *y* and *r* as $x^2 + y^2 = r^2$. Ask students if the relationship represented by this equation would still be true if the point *(x, y)* were in the second quadrant? the third quadrant? The fourth quadrant? Would the equation be true for any point on the circle? Why? Guide students to understand that because every point on the circle makes the equation a true statement and because every point that makes the equation true is on the circle, the equation is the equation of the circle itself. The equation $x^2 + y^2 = r^2$ represents a circle with center at the origin and radius *r*.

Ask students to draw a similar diagram of a circle with center *(h, k)* and radius *r*. Give them time to determine the equation of this circle. Some students may see that the circle is simply shifted *h* units horizontally and *k* units vertically and use transformations to write $(x - h)^2 + (y - k)^2 = r^2$. Others may use the Pythagorean Theorem. Make sure that both methods are discussed.
Expand $(x - h)^2 + (y - k)^2 = r^2$ and discuss the **general** versus the **standard form** of an equation of a circle. Ask students why we might want the equation of a circle written in standard versus general form.

Worktime: Students should work in pairs to complete *Items 1 - 5* of the task.

Give students time to complete and discuss *Item 1*.

Items 2 – 4 provide students with their *first* opportunity to complete the square on a quadratic expression. Remind them that they learned to square binomials when addressing special products in Math I. Give them time to complete *Items 2* and *3* and then ask what algebraic procedure might be used to complete the square on each of the expressions.
Teach a mini-lesson on completing the squares in *x* and *y* to convert from general to standard form of an equation of a circle. Emphasize the fact that values added to the left-hand side of the equation, in order to accomplish completing the squares, must be added to the right-hand side of the equation as well.

Allow students to complete *Item 4* and then share their responses.

In order to graph a circle using the *TI 83/84* calculator, the circle must be entered as two functions. Discuss this process and then allow students to complete *Item 5*.

Closing: Allow students to share their responses to any items not already discussed. Create an anchor chart containing the general form of an equation of a circle and standard forms of equations of circles with radii *r* and centers at the origin and at the point *(h, k)*.

Homework: See attached.

Differentiated support/enrichment:

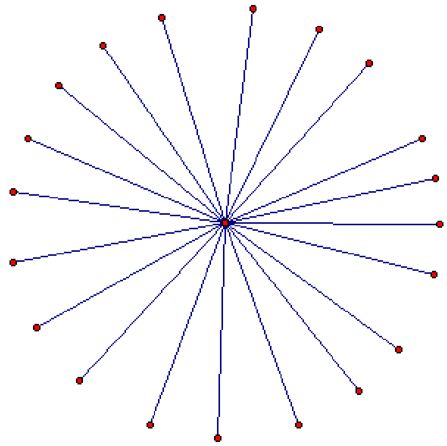
Check for understanding:
Write an equation of a circle in standard form with center (-2, 3) and radius $\sqrt{5}$.

Resources/materials for Math Support: Students should preview:

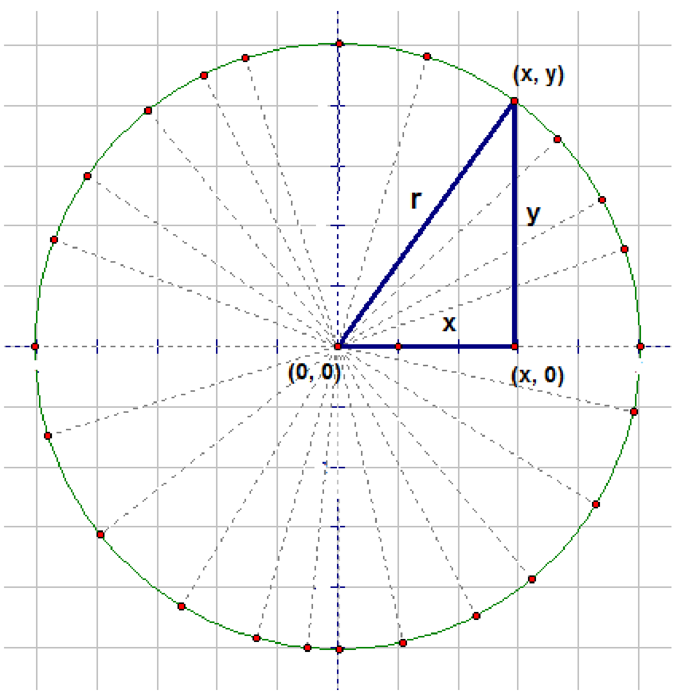
- definition of a circle
- Pythagorean Theorem
- transformations of functions, including horizontal and vertical shifts
- squaring binomials
- completing the square on a quadratic expression

CCGPS Mathematics III
Task 1: ***The Algebra of Circles***
Day 1 Student Task

By definition a circle is the set of all points on a plane equidistant (radius) from a given point (center).



If this circle is drawn on a coordinate plane, with the center located at *(0, 0)* with radius *r*, it is possible to write an algebraic equation for the circle.

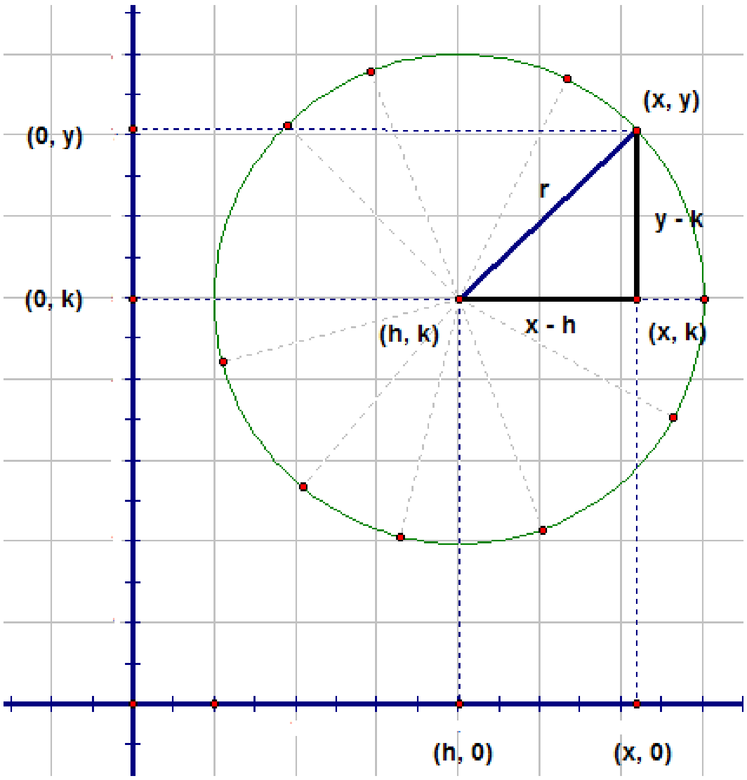


Suppose the center point is located at the origin ***(0, 0)***. Choosing one point ***(x, y)*** allows us to form a right triangle with radius ***r*** as the hypotenuse. One leg is the perpendicular segment from ***(x, y)*** to the x-axis at point ***(x, 0)***. The second leg is the segment from the point ***(x, 0)*** back to the origin. Using the Pythagorean Theorem to write an equation for ***r***, we get

$$x^2 + y^2 = r^2.$$

Because this equation is true for **every** point on the circle, it is considered the equation of the circle itself.

Now suppose the center point is located away from the origin at point ***(h, k)***. Following the procedure used with a circle located with its center at the origin, pick a point ***(x, y)*** on the circle and form a right triangle with the other vertices at ***(x, k)*** and ***(h, k)***.



The length of the hypotenuse is r and the lengths of the legs are $y - k$ and $x - h$.

According to the Pythagorean Theorem,

$r^2 = (x - h)^2 + (y - k)^2$.

By expanding the binomial terms this equation can also be written as

$r^2 = x^2 - 2hx + h^2 + y^2 - 2ky + k^2$

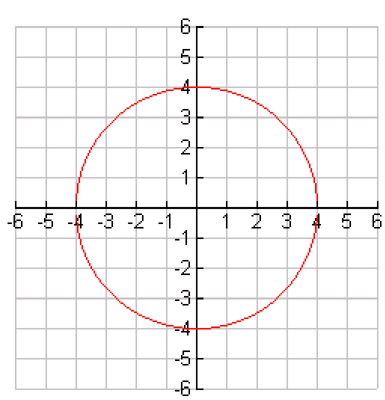
or

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$.

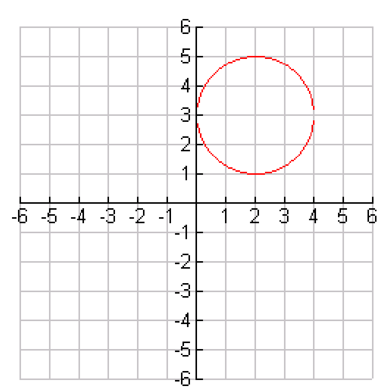
$(x - h)^2 + (y - k)^2 = r^2$ is called the standard form of an equation of a circle with center at ***(h, k)*** and radius r. By multiplying and collecting terms, the standard form of the equation can be written in general form as $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$.

1. Write equations for the following circles in both standard form and general form.

a.



b.



To change from general form of the equation of a circle to standard form, it is necessary to complete the square for x and y. **Completing the square** is an algebraic tool used to change equations in the general form, $Ax^2 + By^2 + Cx + Dy + E = 0$, to standard form,

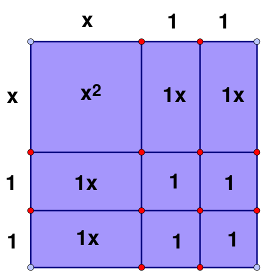
$\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$. Standard form is used to graph circles and other conic sections which

you will study in Mathematics IV.

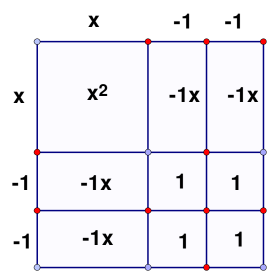
Perfect squares are numbers or expressions which have exactly two identical factors.

$4 = (2)(2)$ $25 = (-5)(-5)$ $9x^2 = (3x)(3x)$ $36y^2 = (-6y)(-6y)$ $x^2 + 4x + 4 = (x + 2)(x + 2)$

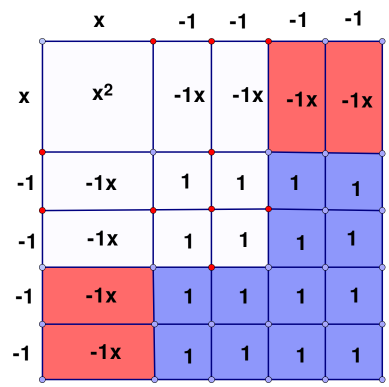
Consider the following area models of three perfect squares. The area is given below each figure in both factored and expanded form.



area = $(x + 2)(x + 2)$
= $x^2 + 4x + 4$



area = $(x - 2)(x - 2)$
= $x^2 - 4x + 4$



area = $(x - 4)(x - 4)$
= $x^2 - 8x + 16$

2. Find the products of the following expressions.

   a. $(x + 1)^2 = (x + 1)(x + 1) =$

   b. $(x - 3)^2 = (x - 3)(x - 3) =$

   c. $(x - 5)^2 = (x - 5)(x - 5) =$

   d. $(x + 7)^2 = (x + 7)(x + 7) =$

   e. $(x + n)^2 = (x + n)(x + n) =$

3. Use the opportunity to review squares of binomials, provided in *Item 2,* to help you complete each of the squares below, showing results in both expanded and factored forms. Include area models to illustrate your answers.

   a. $x^2 + 20x +$ ____ $= (x +$ ____$)^2$

   b. $x^2 - 12x +$ ____ $= (x -$ ____$)^2$

   c. $x^2 + 18x +$ ____ $= (x +$ ____$)^2$

   d. $x^2 - 7x +$ ____ $= (x -$ ____$)^2$

   e. $x^2 + 2nx +$ ____ $= (x +$ ____$)^2$

It is easier to graph a circle when the equation of the circle is given in *standard* rather than *general* form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form, it is necessary to complete the square for both x and y.

$x^2 + y^2 + 2x - 4y - 11 = 0$

$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$ — Group the x and y terms.

$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$ — Complete the square on x and y. Then **balance** the equation by adding the same values to the right side of the equation that were added to the left side (in this case, 1 and 4).

$(x + 1)^2 + (y - 2)^2 = 16$ — Factor. The standard form indicates that this equation represents a circle with center at (-1, 2) and radius 4

The general form of a circle is given as $Ax^2 + By^2 + Cx + Dy + E = 0$, where A and B are always equal. If A and B are some number other than 1, it is necessary to divide both sides of the equation by the value of A (or B) before completing the square.

To change $2x^2 + 2y^2 - 4x + 6y - 4 = 0$ to standard form, divide both sides of the equation by 2 before completing the square for both x and y.

$2x^2 + 2y^2 - 4x + 6y - 4 = 0$	
$x^2 + y^2 - 2x + 3y - 2 = 0$	Divide by 2.
$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$	Group the x and y terms.
$(x^2 - 2x + 1) + (y^2 + 3y + \frac{9}{4}) = 2 + 1 + \frac{9}{4}$	Complete the square on x and y. Balance the equation by adding 1 and $\frac{9}{4}$ to both sides of the equation.
$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$ $(x - 1)^2 + (y + 1.5)^2 = 5.25$	Factor. The standard form indicates that this equation represents a circle with center $(1, -\frac{3}{2})$ and radius $\frac{\sqrt{21}}{2}$. In decimal form, the center is (1, -1.5) and the radius is approximately 2.29.

4. Write the following equations in standard form. Graph the circles and state both the centers and the radii.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 - 4y = 0$

c. $x^2 + y^2 - 6x - 10y = 2$

d. $4x^2 + 16x + 4y^2 - 28y + 25 = 0$

To graph a circle using a *TI83/84* calculator, it is necessary to solve for *y* after changing the equation to standard form.

Consider the following example:

$$x^2 + y^2 + 2x - 4y - 11 = 0.$$

$$(x + 1)^2 + (y - 2)^2 = 16$$

$$(y - 2)^2 = 16 - (x + 1)^2$$

$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$

$$y - 2 = \pm\sqrt{16-(x+1)^2}$$

$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions $y_1 = 2 + \sqrt{\left(16-(x+1)^2\right)}$ and $y_2 = 2 - \sqrt{\left(16-(x+1)^2\right)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with an *x* to *y* ratio of *3* to *2* or the *ZSQUARE* feature. Otherwise circles appear as ellipses.

5. Write the equations from *Item 4* as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 - 4y = 0$

c. $x^2 + y^2 - 6x - 10y = 2$

d. $4x^2 + 16x + 4y^2 - 28y + 25 = 0$

CCGPS Mathematics III
Task 1: *The Algebra of Circles*
Day 1 Homework

Give the center and radius of the following circle.

1. $(x - 2)^2 + (y + 4)^2 = 24$

Write each of the following equations in standard form and graph the circle, stating both center and radius.

2. $x^2 + y^2 + 6x - 8y = 0$
3. $3x^2 + 3y^2 + 6x + 15y + 9 = 0$

4. The diameter of a circle has endpoints (– 2, 4) and (6, 4). Write the equation of the circle in standard form and give its center and radius.

CCGPS Mathematics III
Task 1: The Algebra of *Circles* **Day 2/3**
(GaDOE Task: *Circles and Radio Stations Learning Task (Part 2: Items 1-3)*

CCSS Standard(s):
Algebra
Reasoning with Equations and Inequalities A-REI
Solve systems of equations

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ and the circle $x^2 + y^2 = 3$.

Represent and solve equations and inequalities graphically
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Geometry
Expressing Geometric Properties with Equations G-GPE
Use coordinates to prove simple geometric theorems algebraically
4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*

GPS Standard(s):
MM3G1. Students will investigate the relationships between lines and circles.
a. Find equations of circles.
b. Graph a circle given an equation in general form.
d. Solve a system of equations involving a circle and a line.

New vocabulary:

Mathematical concepts/skills:
- solving systems of equations involving a circle and a line algebraically, graphically, and using appropriate technology
- writing systems of equations involving circles and lines that satisfy given conditions (for example, a circle and a line that have no points in common)

Prior knowledge:
- solving systems of linear equations in two variables algebraically and graphically

Essential question(s): How can I determine where a circle and a line intersect?

Suggested materials:
- graph paper

- graphing calculators

Warm-up: Post the following:

1. *How many times can a circle and a line intersect? Illustrate your response with a sketch.*
2. *What does your response to Question 1 tell you about the solution set of a system of equations involving a circle and a line?*

Opening: Allow students to share their responses to the *Warm-up*. A circle and a line can intersect 0, 1, or 2 times. This means that the solution set of a system of equations that involves a circle and a line may contain 0, 1, or 2 ordered pairs.

Give students an opportunity to complete *Item 8*. Then teach a mini-lesson on solving systems of equations involving a line and a circle by allowing students to share responses to this item.

Verify solutions to the system in *Item 8* by allowing students to complete and share *Item 9*.

Worktime: Students will have completed *Items 6 – 9* during the *Warm-up* and the *Opening*. They should work in pairs to complete *Items 10* and *11*.

It is reasonable to allow students to complete 3 or 4 of the 5 systems in *Item 10* and assign the remaining systems for homework. Allow students to share solutions to 2 or 3 of these systems before beginning *Item 11*.

Closing: Allow students to share their responses to *Item 11*.

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

- solving systems of linear equations algebraically
- solving systems of quadratic equations

CCGPS Mathematics III
Task 1: *The Algebra of Circles*
Day 2 Student Task

In this part of our task on circles, we will learn to solve systems of equations that contain an equation of a circle and an equation of a line.

6. How many times can a circle and a line intersect? Illustrate your response with a sketch.

7. What does your response to *Item 1* tell us about the solution set of a system of equations involving a circle and a line?

We can solve systems of equations involving a circle and a line by substitution and by graphing.

8. The system $\begin{cases} x^2 + y^2 = 9 \\ x + y = 1 \end{cases}$ is solved below. Justify each step used to find the solution of the system.

$y = -x + 1$

$x^2 + (-x + 1)^2 = 9$

$x^2 + x^2 - 2x + 1 = 9$

$2x^2 - 2x - 8 = 0$

$x^2 - x - 4 = 0$

$x = \frac{1 \pm \sqrt{17}}{2}$, which gives

$x = 2.56$ and $x = -1.56$

$\{(2.56, -1.56), (-1.56, 2.56)\}$

9. Use your graphing calculator to graph the three functions needed to represent the system given in *Item 8*. Then use the *CALC/intersect* feature of the calculator to verify the solution set of the system.

10. Solve the following systems of equations algebraically and then check the solutions using your graphing calculator.

a. $\begin{cases} x^2 + y^2 = 34 \\ x - y = 2 \end{cases}$

b. $\begin{cases} x^2 + y^2 = 9 \\ 2y = x + 8 \end{cases}$

c. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

d. $\begin{cases} x^2 + y^2 = 25 \\ -3x + 4y = 25 \end{cases}$

e. $\begin{cases} x^2 + y^2 = 10 \\ x + 3y = 10 \end{cases}$

11. Write a system of equations that satisfies each of the conditions given below. In each case, verify your work by solving the system algebraically and by graphing.
    a. a line and a circle that have no points in common
    b. a line and a circle that are tangent
    c. a line and a circle that have two points of intersection

CCGPS Mathematics III
Task 1: *The Algebra of Circles* **Day 3/3**
(GaDOE Task: *Circles and Radio Stations; Part 2, Items 4 and 5)*

CCSS Standard(s):
Algebra
Reasoning with Equations and Inequalities A-REI
Understand solving equations as a process of reasoning and explain the reasoning
1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve systems of equations
5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Represent and solve equations and inequalities graphically
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

GPS Standard(s):
MM3G1. Students will investigate the relationships between lines and circles.
a. Find equations of circles.
b. Graph a circle given an equation in general form.
c. Find the equation of a tangent line to a circle at a given point.
d. Solve a system of equations involving a circle and a line.
e. Solve a system of equations involving two circles.

New vocabulary: point of tangency

Mathematical concepts/skills:
- deriving the equation of a line tangent to a circle at a given point
- writing the equation of a circle tangent to a given line
- writing systems of equations involving circles and lines that satisfy given conditions (for example, a circle and a line that are tangent at a given point)
- solving systems of equations involving a circle and a line algebraically, graphically, and using appropriate technology
- solving systems of equations involving two circles

Prior knowledge:
- writing the equation of a line given characteristics of the line (for example, given one point and the slope)
- properties of circles and their tangent lines
- solving systems of linear equations in two variables algebraically and graphically

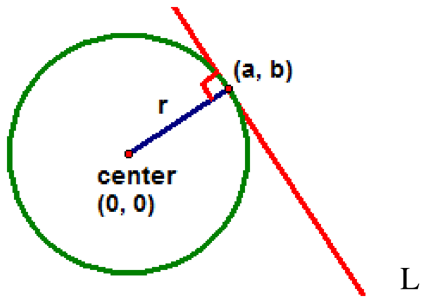
Essential question(s): How can I find the equation of a line tangent to a circle at a given point? How can I find where two circles intersect?

Suggested materials:

- graph paper
- graphing calculators

Warm-up: Post the following:

Write an equation for the line L. (Hint: What do you know about the relationship between the radius and a tangent line to a circle at the point of tangency?)



Opening: Students should remember that a radius of a circle is perpendicular to a tangent line at the point of tangency. The derivation of the equation $y - b = \frac{-a}{b}(x - a)$ is given in *Item 12* of the task. Guide students to an understanding of this equation by questioning and using their responses to the *Warm-up*.

Give students time to complete *Item 12*, justifying the steps taken to derive the equivalent equation $ax + by = r^2$. Discuss the steps. Make sure that all students understand that this is the equation of a line tangent to a circle, with center at the origin and radius *r*, at the point (*a, b*).

Worktime: Students should work in pairs to complete the task.

Allow time for students to complete *Items 13* and *14* of the task and then discuss responses.

Allow time for students to complete *Item 15* and discuss responses. Then teach a mini-lesson on solving systems of equations involving two circles, using the example provided in the task. Allow students to verify the solution set of this system using their graphing calculators (*Item 16*) before proceeding to *Item 17*.

Closing: Allow students to share their responses to *Item 17*.

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

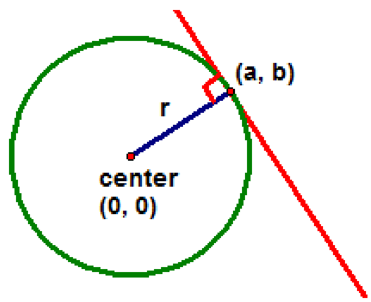
- writing the equation of a line given characteristics of the line (for example, given one point and the slope)
- properties of circles and their tangent lines
- solving systems of two linear equations algebraically using the elimination method

CCGPS Mathematics III
Task 1: *The Algebra of Circles*
Day 3 Student Task

In the previous lesson, you were asked to write a system of equations involving a circle and a line tangent to the circle. Chances are you accomplished this by drawing a graph and/or making a guess and then verifying that guess. It is possible to find the equation of any line tangent to a given circle if you know the point of tangency.

From our study of circles in Math II, we know that a tangent line intersects a circle in exactly one point called the *point of tangency*. We also know that a radius drawn to the point of tangency is perpendicular to the tangent.

In the following diagram, the **point of tangency** is ***(a, b)*** and the **radius** is ***r***. The equation of this circle is $x^2 + y^2 = r^2$ and by substituting *(a, b)* for *(x, y)* we get $a^2 + b^2 = r^2$.



In order to write the equation of a line, we need a point on the line and the slope. In this case, we know the slope of the radius is $\frac{b}{a}$. Since the tangent line is perpendicular to the radius, the slope of the tangent line must be $\frac{-a}{b}$. (*a*, *b*) is a point on both the radius and the tangent line. Substituting the point *(a, b)* and $m = \frac{-a}{b}$ into the point-slope form of an equation of a line gives

$$y - b = \frac{-a}{b}(x - a).$$

12. The equation $ax + by = r^2$ is the equation of a line tangent to a circle, with center (0,0) and radius *r*, at the point *(a, b)*. Justify the steps used below to arrive at this equation from the point-slope form of the equation determined above.

$$y - b = \frac{-a}{b}(x - a)$$

$$b(y - b) = -a(x - a)$$

$$by - b^2 = -ax + a^2$$

$$ax + by = a^2 + b^2$$

$$\boldsymbol{ax + by = r^2}$$

13. Find an equation of the tangent line to a circle with the equation $x^2 + y^2 = 9$ with the point of tangency at $(1, \sqrt{8})$.

14. Write the equation of a circle with the center at the origin tangent to the line $2x + 3y = 13$.

We will now focus our attention on solving systems of equations consisting of equations of two circles. These systems can be solved using elimination, substitution, or graphing.

15. How many different solutions are possible when solving a system of equations representing two circles? Use diagrams to explain your answer.

Consider the process for solving the following system of equations:

$$\begin{cases} x^2 + (y-2)^2 = 9 \\ (x-2)^2 + (y-3)^2 = 1 \end{cases}.$$

$$\begin{cases} x^2 + y^2 - 4y + 4 = 9 \\ x^2 + y^2 - 4x - 6y + 13 = 1 \end{cases}$$

Expand the binomials.

$$\begin{cases} x^2 + y^2 - 4y - 5 = 0 \\ x^2 + y^2 - 4x - 6y + 12 = 0 \end{cases}$$

Collect like terms, setting both equations equal to zero.

$4x + 2y - 17 = 0$	Subtract the two equations to obtain a linear equation. (Note: Any solutions of the system must lie on this line.)
$y = -2x + 8.5$	Solve for y in terms of x to get.
$x^2 + (\mathbf{-2x + 8.5} - 2)^2 = 9$	Substitute $-2x + 8.5$ in place of y in either of the original equations.
$x^2 + (-2x + 6.5)^2 = 9$	Simplify.
$x^2 + 4x^2 - 26x + 42.25 - 9 = 0$	Expand the binomial and simplify.
$5x^2 - 26x + 33.25 = 0$	Solve for x using the Quadratic Formula.
$x = \dfrac{26 \pm \sqrt{11}}{10}$ or $x \approx 2.9$ or $x \approx 2.2$	
$\{(2.9, 2.7), (2.3, 3.9)\}$	Substitute $x = 2.9$ and $x = 2.3$ into the linear equation to obtain approximate solutions.

16. Use your graphing calculator and the CALC/intersect feature to verify the solutions of the system above.

17. Solve each of the following systems of equations algebraically. Then verify your results using your graphing calculator.

a. $\begin{cases} x^2 + y^2 = 9 \\ (x-1)^2 + y^2 = 4 \end{cases}$

b. $\begin{cases} x^2 + y^2 = 4 \\ (x-1)^2 + (y+2)^2 = 9 \end{cases}$

CCGPS Mathematics III
Task 1: *The Algebra of Circles*
Day 3 Homework

A circle has center at the origin and a radius of 4.

1. Write an equation of the line tangent to the circle at the point in the first quadrant where $x = 2$.

2. Verify your work in Question 1 by writing and solving the system of equations that includes the equation of the indicated circle and the equation of the tangent line that you derived.

Solve each of the following systems of equations.

3. $\begin{cases} x^2 + y^2 = 4 \\ (x-1)^2 + y^2 = 4 \end{cases}$

4. $\begin{cases} (x-1)^2 + y^2 = 9 \\ x^2 + (y-2)^2 = 4 \end{cases}$

Applying the Algebra of Circles to Real Situations

The following tasks provide excellent opportunities for students to apply what they have learned about circles to real situations. They may be used as deemed appropriate.

Mini-task 1: Radio Stations

1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.

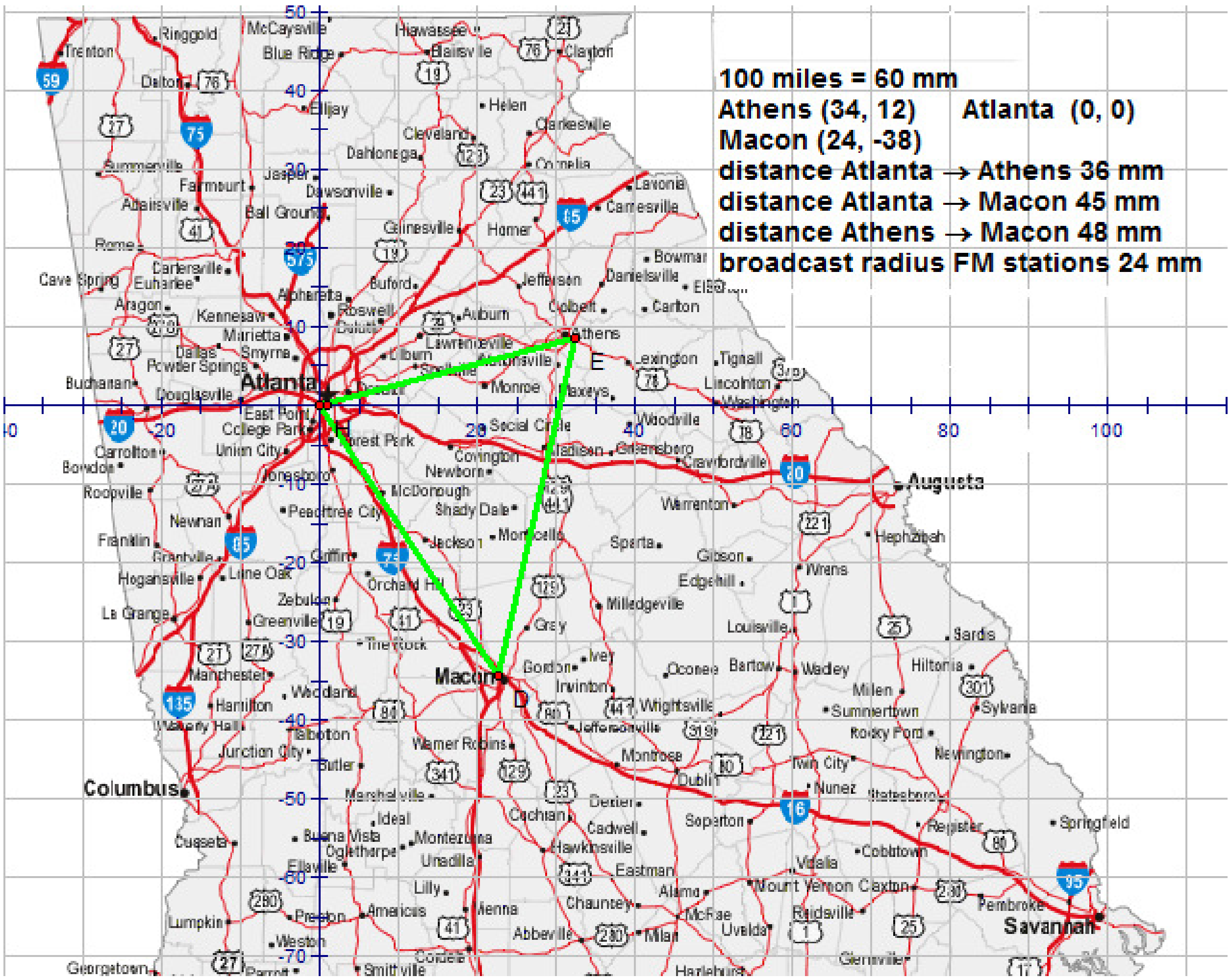
   a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area on the partial map of Georgia given on the following page. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale of 100 miles = 60 mm.
   b. Find an equation which represents the station's maximum listening area.
   c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use your map to help you answer the following questions.

   a. Given the location of Randy's house, can he expect to pick up radio signals from WXAG and WDEN? Show how you know.

   b. What are the coordinates of the intersections of the broadcast areas of station WYAY and station WDEN? Show your work. (Does it matter whether you find the intersections using miles or mms? Explain.)

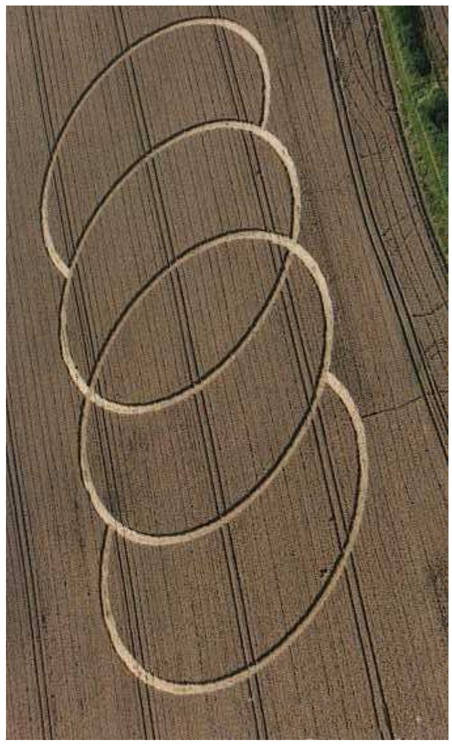
Partial Map of Georgia



Mini-task 2: Crop Circles

Crop circles, geometric patterns formed by flattening grain crops, have been documented since the 17th century in English woodcuts. Early crop circles were simple circular formations, but more recent formations have increased in complexity and now include figures other than circles. Many crop circles are extremely large and are best viewed from the air. Their size and precision of patterns, added to the fact that many appear mysteriously overnight have fueled interest in this phenomenon. A large number of crop circles began appearing in the English countryside during the 1970s and were reported in Australia and the United States in the 1980s. Many crop circles are known to be man-made, but a large number have unexplained origins. For more information on crop circles visit, *http://en.wikipedia.org/wiki/Crop_circle*

More recent crop circles have intricate patterns, similar to computer graphics. Choose one of the crop circles shown below (or a pattern approved by your teacher) and develop equations to generate similar designs on your calculator or computer.



ATLANTA PUBLIC SCHOOLS
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Task 2: Parabolas

CCGPS Mathematics III
Task 2: *Parabolas* **Day 1/2**
(GaDOE Task: *Parabolas Learning Task, Items 1-5*)

CCSS Standard(s):
Algebra
Reasoning with Equations and Inequalities A-REI
Represent and solve equations and inequalities graphically
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Geometry
Expressing Geometric Properties with Equations G-GPE
Translate between the geometric description and the equation for a conic section
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically
4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*

GPS Standard(s):
MM3G2. Students will recognize, analyze, and graph the equations of the conic sections (parabolas and circles).
b. Graph conic sections, identifying fundamental characteristics.
c. Write equations of conic sections given appropriate information.

New vocabulary: focus, directrix

Mathematical concepts/skills:
- identifying the vertex, focus, directrix, and axis of symmetry of a parabola, given its equation
- graphing a parabola given its equation
- writing the equation of a parabola given characteristics of its graph

Prior knowledge:
- graphing quadratic functions written in standard or vertex form

Essential question(s): How can I graph a parabola given its equation? How can I write an equation of a parabola given characteristics of its graph?

Suggested materials:
- graph paper

- graphing calculators

Warm-up: Post the following:

Write the following quadratic equation in vertex form and draw its graph.

$$y = -2x^2 + 12x - 14$$

Opening: Discuss student responses to the *Warm-up*. Then give students time to read the opening paragraphs of the task down to the table summarizing information for equations of parabolas.

Teach a mini-lesson that includes the following:
- geometric definition of a parabola
- how to determine from an equation whether a parabola is vertical or horizontal
- how to identify the vertex, focus, directrix, and axis of symmetry from the equation of a parabola written in standard form
- how to graph a parabola given its equation in standard form

Create an anchor chart containing standard forms of equations of parabolas and information needed to determine vertex, focus, directrix, and axis of symmetry.

Worktime: Students should work in pairs to complete *Items 1 - 3* of the task.

Discuss all problems in *Item 1* before allowing students to begin *Items 2* and *3*.

Closing: Allow students to share their responses to *Items 2* and *3*.

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview graphing quadratic functions written in standard and in vertex form.
:

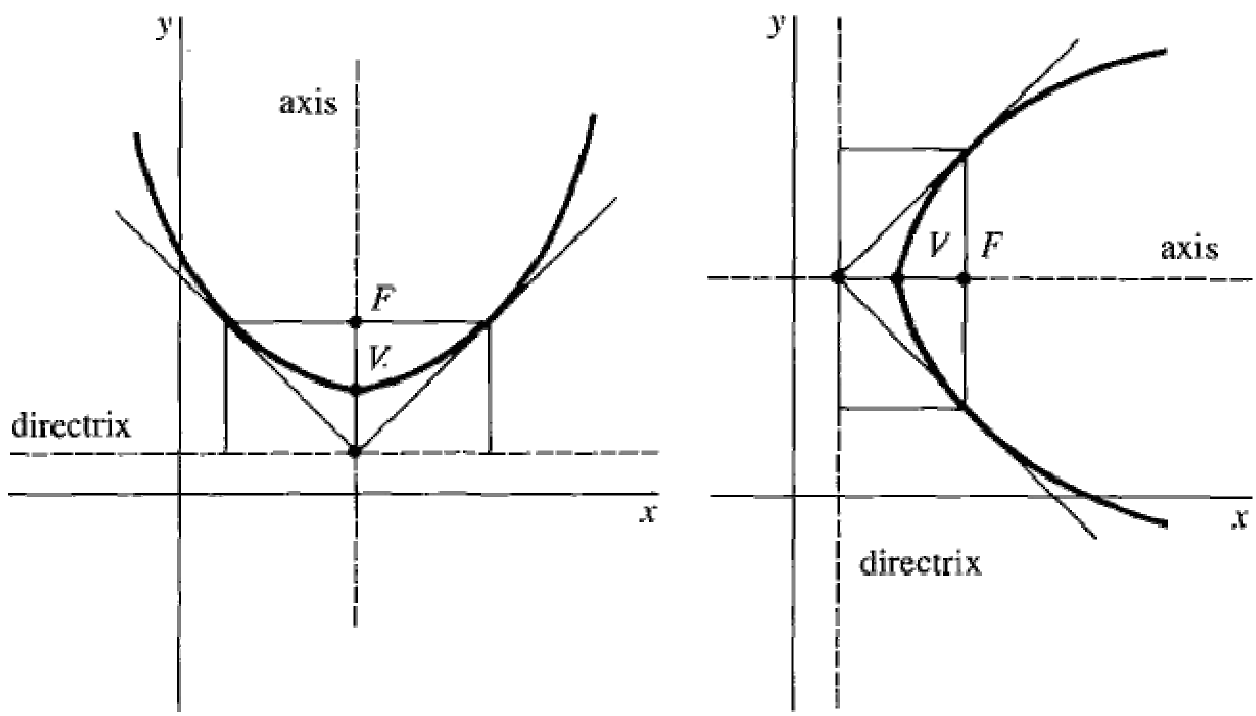
CCGPS Mathematics III
Task 2: ***Parabolas***
Day 1 Student Task

Parabolas were studied in Math I and Math II as graphs of quadratic functions. Every parabola that is the graph of a function is a vertical parabola, meaning it opens either up or down. However, not all parabolas are vertical. They may also be horizontal, meaning they open to the left or to the right. Consider the pictures below. Could a horizontal parabola be the graph of a function? Why or why not?

The geometric definition of a parabola (vertical or horizontal) is as follows:

A parabola is the set of all points in a plane equidistant from a given point, called the **focus**, and a given line, called the **directrix**.

Consider the diagrams below.



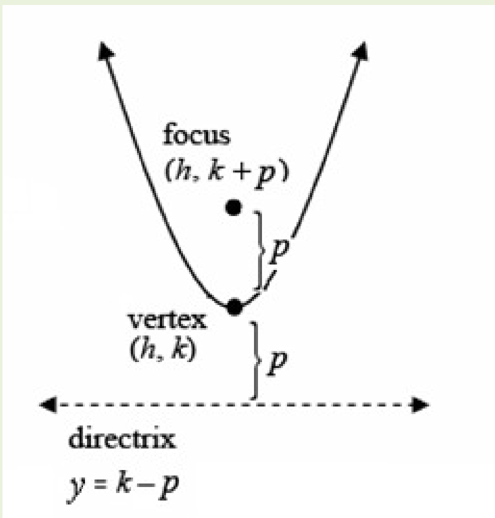
(a) Vertical (b) Horizontal

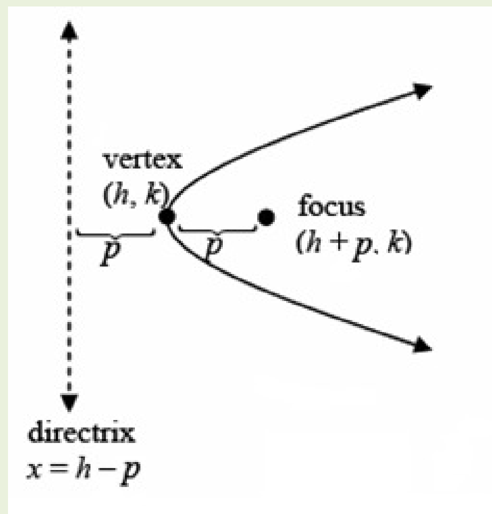
In both diagrams, *F* represents the focus, *V* represents the vertex, and the dotted line represents the directrix of the parabola. Every point on a parabola is equidistant from its focus and its directrix.

The general form of an equation of a parabola that is either horizontal or vertical is $Ax^2 + By^2 + Dx + Ey + F = 0$, where either $A = 0$ or $B = 0$ but not both A and B can be 0. From past experience with quadratic functions, you might have guessed that if the coefficient of y^2 or B is equal to 0, then the parabola is vertical. If the coefficient of x^2 or A is equal to 0, then the parabola is horizontal.

The table below shows the standard form for equations of horizontal and vertical parabolas and how to find the vertex, focus, and directrix given these forms.

Summary of Information on Parabolas





$$Ax^2 + By^2 + Dx + Ey + F = 0$$

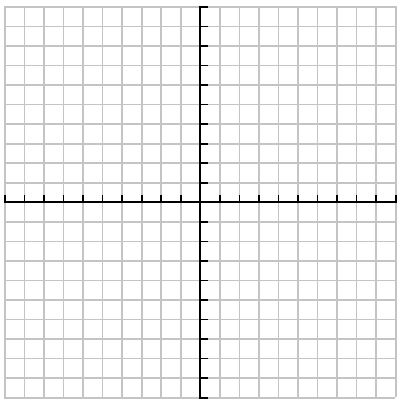
where $A = 0$ or $B = 0$ but not both A and B are 0

vertex at $(h, \mathbf{k})$

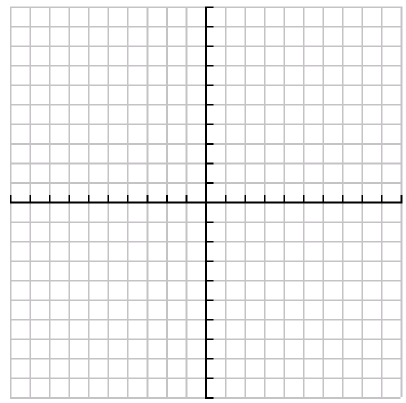
Vertical Parabola (horizontal directrix)	Horizontal Parabola (vertical directrix)
$y - k = \frac{1}{4\text{p}}(x - h)^2$	$x - h = \frac{1}{4\text{p}}(y - k)^2$
$\text{p} > 0$ opens up	$\text{p} > 0$ opens right

1. For each of the parabolas below, list the coordinates of the vertex, the coordinates of the focus, the equation of the directrix, and the axis of symmetry. Then graph the parabola.

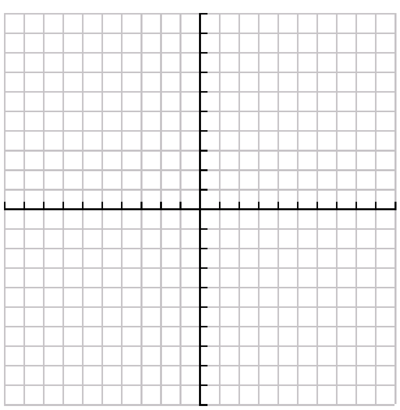
a. $(y - 3)^2 = -12(x + 2)$



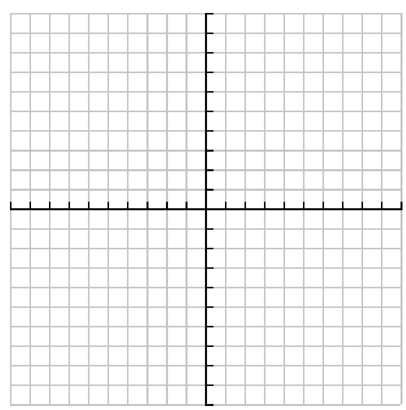
b. $-\frac{1}{8}(x - 1)^2 = y$



c. $x = (y - 4)^2$



d. $(x + 1)^2 = 2(y + 3)$



2. Write an equation of the parabola with focus (3, 5) and directrix $y = -1$.

3. Write an equation of the parabola with focus (2, 3) and directrix $x = 6$.

CCGPS Mathematics III
Task 2: *Parabolas* **Day 2/2**
(GaDOE Task: *Parabolas Learning Task, Items 2 - 4)*

CCSS Standard(s):
Algebra
Reasoning with Equations and Inequalities A-REI
Represent and solve equations and inequalities graphically
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Geometry
Expressing Geometric Properties with Equations G-GPE
Translate between the geometric description and the equation for a conic section
2. Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically
4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*

GPS Standard(s):
MM3G2. Students will recognize, analyze, and graph the equations of the conic sections (parabolas and circles).
a. Convert equations of conics by completing the square.
b. Graph conic sections, identifying fundamental characteristics.
c. Write equations of conic sections given appropriate information.

New vocabulary:

Mathematical concepts/skills:
- converting from general to vertex form of an equation of a parabola by completing the square
- identifying the vertex, focus, directrix, and axis of symmetry of a parabola, given its equation
- graphing a parabola given its equation

Prior knowledge:
- graphing quadratic functions written in standard or vertex form

Essential question(s): How can I graph a parabola given its equation?

Suggested materials:
- graph paper
- graphing calculators

Warm-up: Post the following:

Complete the square on the following equation.

$$y = -2x^2 + 12x - 14$$

Use what you learned in the previous lesson to state the vertex, focus, directrix, and axis of symmetry of the parabola represented by this equation.

Opening: Discuss student responses to the *Warm-up.*

Teach a mini-lesson on completing the square on a parabolic equation using the Warm-up and the example provided in the task.

Worktime: Students should work in pairs to complete *Items 4 - 6* of the task.

Closing: Allow students to share their responses to *Items 4 -6*.

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview graphing quadratic functions written in standard and in vertex form.
:

CCGPS Mathematics III
Task 2: ***Parabolas***
Day 2 Student Task

As with circles, we often change equations of parabolas from general form to standard form in order to facilitate graphing. To do this, we must complete the square.

Consider the equation $2x^2 - 4x + y + 4 = 0$.

We first rewrite the equation with x terms and y terms on different sides of the equation.

$$2x^2 - 4x + y + 4 = 0$$

$y + 4 = -2x^2 + 4x$ — separate *x* terms and y terms

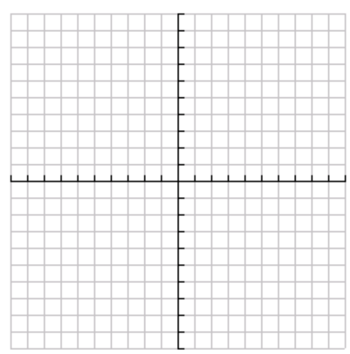
$y + 4 = -2\left(x^2 - 2x\right)$ — prepare to complete the square by factoring out -2

$y + 4 - 2 = -2\left(x^2 - 2x + 1\right)$ — complete the square; add -2 to both sides

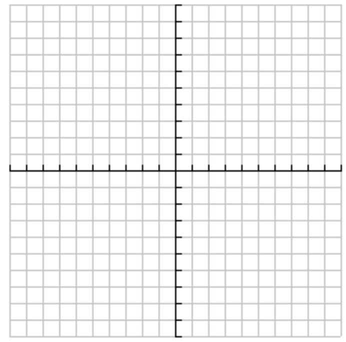
$y + 2 = -2\left(x - 1\right)^2$ — factor

4. Write each of the following equations in standard form. List the vertex, coordinates of the focus, equation of the directrix and the axis of symmetry. Graph the parabola.

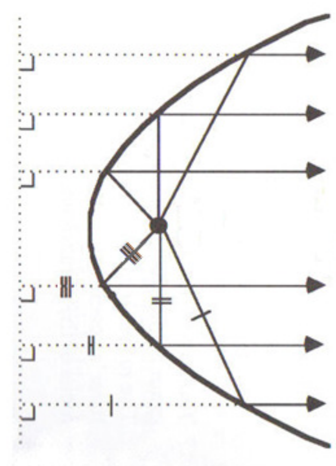
a. $y^2 - 8y + 8x + 8 = 0$



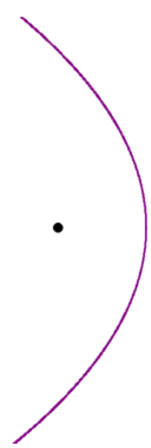
b. $x^2 - 6x + 12y + 21 = 0$



Parabolas have special reflective properties which make them useful shapes for many items including flashlights, car headlights, suspension bridges, solar cookers, and satellite dishes. Two properties are especially important when considering applications of parabolas. First, all rays in the interior of a parabola parallel to the axis of symmetry are reflected toward the focus. And, all rays emitted from the focus are reflected so that each reflected ray runs parallel to the axis of symmetry and perpendicular to the directrix.



5. Approximate the location of the focus point and the directrix on the following parabola. Show how rays emitted from the focus would travel.



6. Parabolas and Suspension Bridges.

Suspension bridges depend on parabolically curved cables to support the weight of the road bed of the bridge. The weight of the bridge is evenly distributed among the support cables which run parallel to the axis of symmetry, similiar to the paths of rays reflected off the surface of parabolic reflectors.

The Golden Gate bridge is a suspension bridge in San Francisco, California. The towers are 1280 meters apart and rise 160 meters above the road. The cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower?