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Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 3
Circles and Spheres

GE Foundation

This document has been made possible by funding from the GE Foundation Developing Futures grant, in partnership with Atlanta Public Schools. It is derived from the Georgia Department of Education Math II Framework and includes contributions from Georgia teachers. It is intended to serve as a companion to the GA DOE Math II Framework Teacher Edition. Permission to copy for educational purposes is granted and no portion may be reproduced for sale or profit.

Preface

We are pleased to provide this supplement to the Georgia Department of Education's Mathematics II Framework. It has been written in the hope that it will assist teachers in the planning and delivery of the new curriculum, particularly in these first years of implementation. This document should be used with the following considerations.

- The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics teachers should work the task, read the teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

- This guide provides day-by-day lesson plans. While a detailed scope and sequence and established lessons may help in the implementation of a new and more rigorous curriculum, it is hoped that teachers will assess their students informally on an on-going basis and use the results of these assessments to determine (or modify) what happens in the classroom from one day to the next. Planning based on student need is much more effective than following a pre-determined timeline.

- It is important to remember that the Georgia Performance Standards provide a balance of concepts, skills, and problem solving. Although this document is primarily based on the tasks of the Framework, we have attempted to help teachers achieve this all important balance by embedding necessary skills in the lessons and including skills in specific or suggested homework assignments. The teachers and writers who developed these lessons, however, are not in your classrooms. It is incumbent upon the classroom teacher to assess the skill level of students on every topic addressed in the standards and provide the opportunities needed to master those skills.

- In most of the lesson templates, the sections labeled *Differentiated support/enrichment* have been left blank. This is a result of several factors, the most significant of which was time. It is hoped that as teachers use these lessons, they will contribute their own ideas, not only in the areas of differentiation and enrichment, but in other areas as well. Materials and resources abound that can be used to contribute to the teaching of the standards.

On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

We hope that teachers will incorporate these strategies in each and every lesson. It is hoped that you find this document useful as we strive to raise the mathematics achievement of all students in our district and state. Comments, questions, and suggestions for inclusions related to the document may be emailed to Dr. Dottie Whitlow, Executive Director, Mathematics and Science Department, Atlanta Public Schools, dwhitlow@atlantapublicschools.us

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Explanation of the Terms and Categories Used in the Lesson Template

Task: This section gives the suggested number of days needed to teach the concepts addressed in a task, the task name, and the problem numbers of the task as listed in the Georgia Department of Education's Mathematics II Framework Teacher Edition (GaDOE TE).

In some cases new tasks or activities have been developed. These activities have been named by the writers.

Standard(s): Although each task addresses many Math II standards and uses mathematics learned in earlier grades, in this section, only the key standards addressed in the lesson are listed.

New Vocabulary: Vocabulary is listed here the *first* time it is used. It is strongly recommended that teachers, particularly those teaching Math Support, use interactive word walls. Vocabulary listed in this section should be included on the word walls and previewed in Math Support.

Mathematical concepts/skills: Major concepts addressed in the lesson are listed in this section whether they are Math II concepts or were addressed in earlier grades or courses.

Prior knowledge: Prior knowledge includes only those topics studied in previous grades or courses. It does not include Math II content taught in previous lessons.

Essential Question(s): Essential questions may be daily and/or unit questions.

Suggested materials: This is an attempt to list all materials that will be needed for the lesson, including consumable items, such as graph paper; and tools, such as graphing calculators and compasses. This list does not include those items that should always be present in a standards-based mathematics classroom such as markers, chart paper, and rulers.

Warm-up: A suggested warm-up is included with every lesson. Warm-ups should be brief and should focus student thinking on the concepts that are to be addressed in the lesson.

Opening: Openings should set the stage for the mathematics to be done during the work time. The amount of class time used for an opening will vary from day-to-day but this should not be the longest part of the lesson.

Worktime: The problem numbers have been listed and the work that students are to do during the worktime has been described. A student version of the day's activity follows the lesson template in every case. In order to address all of the standards in Math II, some of the problems in some of the original GaDOE tasks have been omitted and less time consuming activities have been substituted for those problems. In many instances, in the student versions of the tasks, the writing of the original tasks has been simplified. In order to preserve all vocabulary, content, and meaning it is important that teachers work the original tasks as well as the student versions included here.

Teachers are expected to both facilitate and provide some direct instruction, when necessary, during the work time. Suggestions related to student misconceptions, difficult concepts, and deeper meaning have been included in this section. However, the teacher notes in the GaDOE Math II Framework are exceptional. In most cases, there is no need to repeat the information provided there. Again, it is imperative that teachers work the tasks and read the teacher notes that are provided in GaDOE support materials.

Questioning is extremely important in every part of a standards-based lesson. We included suggestions for questions in some cases but did not focus on providing good questions as extensively as we would have liked. Developing good questions related to a specific lesson should be a focus of collaborative planning time.

Closing: The closing may be the most important part of the lesson. This is where the mathematics is formalized. Even when a lesson must be continued to the next day, teachers should stop, leaving enough time to "close", summarizing and formalizing what students have done to that point. As much as possible students should assist in presenting the mathematics learned in the lesson. The teacher notes are all important in determining what mathematics should be included in the closing.

Homework: In some cases, homework suggestions are provided. Teachers should use their resources, including the textbook, to assign homework that addresses the needs of their students.

Homework should be focused on the skills and concepts presented in class, relatively short (30 to 45 minutes), and include a balance of skills and thought-provoking problems.

Differentiated support/enrichment: On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution;
- flexible grouping of students;
- multiple representations of mathematical concepts;
- writing in mathematics;
- monitoring of progress through on-going informal and formative assessments; and
- analysis of student work.

Check for understanding: A check for understanding is a short, focused assessment-a ticket out the door, for example. There are many good resources for these items, including the GaDOE culminating task at the end of each unit and the *Mathematics II End-of-Course Study Guide*. Both resources can be found on-line at www.georgiastandards.org, along with other GaDOE materials related to the standards. Problem numbers from the GaDOE culminating task have been listed with the appropriate lessons in this document.

Resources/materials for Math Support: Again, in some cases, we have provided materials and/or suggestions for Math Support. This section should be personalized to your students, class, and/or school, based on your resources.

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Task Notes

The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics, teachers should work the task, read any corresponding teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

The tasks provided in this Supplement are based on the content of Unit 3 of the Georgia Department of Education's Mathematics II Framework. Unlike other units of the Atlanta Public Schools Curriculum Supplement, the tasks presented here vary significantly from those in the GaDOE Framework. Where possible, tasks and individual problems of the GaDOE Framework are referenced in order for teachers to take advantage of any teacher notes provided by the GaDOE. We suggest, as always, that teachers use this reference along with the *Mathematics II End-of-Course Study Guide* which can be found on-line at www.georgiastandards.org.

Over and over in this unit of the APS Supplement, students are asked to prove the theorems being discovered. There is obviously not time for students to develop a formal proof, be it two-column, paragraph or flowchart, for every theorem included in the unit. It is up to individual teachers to decide which theorems are to be proven and which are justified in a more informal manner. We do feel it is imperative that students prove enough theorems, particularly those in the first two tasks, to understand that properties of circles are simply applications of triangle congruence and similarity.

Task 1: Relationships among Central Angles, Arcs, and Chords
The big ideas presented in this task include:

- development and proof of the following theorems;
    - In a circle or congruent circles, two chords are congruent if and only if their arcs are congruent.
    - Two chords of a circle are congruent if and only if they are equidistant from the center of the circle.
    - A radius is perpendicular to a chord if and only if it bisects the chord.
    - If a radius is perpendicular to a chord, then it bisects the chord and the arc intercepted by the chord.
    - Any perpendicular bisector of a chord passes through the center of the circle.
- Properties of circles are applications of triangle congruence and similarity.

Activities labeled as *Perpendicular Chord Bisector* and *Circle Center*, which can be found at the Geogebra site http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29, contain discovery and skills practice related to the concepts in task. Other materials can be found on the GaDOE 's *Learning Village*.

Task 2: Investigating Circle Relationships

The big ideas presented in this task include:

- development and proof of the following theorems;
  - The measure of an angle inscribed in a circle is one-half the measure of its intercepted arc.
  - Opposite angles of a quadrilateral inscribed in a circle are supplementary.
  - The measure of an angle formed by two intersecting chords is one-half the sum of the measures of the two intercepted arcs.

Discovery and skills practice related to the concepts in this task can be found at the Geogebra site: http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29 and also on the GaDOE 's *Learning Village*.

Task 3: Sunrise on the First Day of the New Year

The big ideas presented in this task include:

- development and proof of the following theorems;
  - The measure of an angle formed by two secants that intersect outside a circle is one-half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.
  - The measure of an angle formed by a secant and a tangent that intersect outside a circle is one-half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.
  - The measure of an angle formed by the intersection of a tangent and a chord at the point of tangency is one-half the measure of the intercepted arc.
  - The measure of an angle formed by two tangents that intersect outside a circle is one-half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.
- the use of dynamic geometry software in the development and proof of properties of circles.

Parts of this task were adapted from Task 1 of the GaDOE Framework.

Activities labeled as *Two Secants* and *Two Secants Practice*, which can be found at the Geogebra site http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29, contain discovery and skills practice related to the concepts in this task. Other materials can be found on the GaDOE 's *Learning Village*.

Task 4: Investigating Lengths of Segments

The big ideas presented in this task include:

- use of dynamic geometry software to investigate relationships among lengths of segments formed by chords, secants, and tangents;
- verification and proof of the following theorems:
  - If two chords intersect in a circle, the product of the lengths of the segments of one chord equals the product of the segments of the other chord.
  - If two secant segments are drawn to a circle from the same external point, the product of the length of one secant segment and its external part is equal to the product of the length of the other secant segment and its external part.
  - If a secant segment and tangent segment are drawn to a circle from the same external point, the product of the length of the secant segment and its external part equals the square of the length of the tangent segment.
  - Tangent segments drawn to a circle from a point outside the circle are congruent.

The following geogebra websites contain discovery and skills practice related to the concepts in this task.

http://www.geogebra.org/en/upload/files/UC_MAT/chords_in_a_circle.html
http://www.geogebra.org/en/upload/files/UC_MAT/chords_outside_a_circle.html

Task 5: Finding Arc Length and the Area of a Sector
The big ideas presented in this task include:

- finding the length of an arc as a portion of the circumference of a circle
- finding the area of a sector as a portion of the area of a circle

Task 6: Volume and Surface Area of a Sphere
The big ideas presented in this task include:

- derivation and application of formula for volume of a sphere
- derivation and application of formula for surface area of a sphere
- the effect of the change in the radius of a sphere on its surface area and volume

Skills problems related to finding the volume and surface area of a sphere can be found at http://www.geogebra.org/en/upload/files/english/lewisprisco/sphere.html.

Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 3
Circles and Spheres

Task 1: Relationships among Central Angles, Arcs, and Chords

Mathematics II
Task 1: ***Relationships among Central Angles, Arcs, and Chords*** **Day 1/2**
(GaDOE TE Task 1: # 4)

MM2G3. Students will understand the properties of circles.

a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.

b. Understand and use properties of central, inscribed, and related angles.

d. Justify measurements and relationships in circles using geometric and algebraic properties.

New vocabulary: central angle, arc, major arc, minor arc, semicircle, subtend, intercept, arc measure, arc length, chord

Mathematical concepts/skills:

- definitions of concepts including central angle, arc, major arc, minor arc, semicircle, arc measure, arc length, congruent arcs, and chord
- biconditional statements
- proof of the following biconditional statement:
  In the same circle or congruent circles, two chords are congruent if and only if their subtended arcs are congruent.

Prior knowledge:

- definitions of circle, radius, diameter, circumference
- degree measure of a complete circle is 360°
- biconditional statements
- triangle congruence postulates

Essential question(s): What is the basic knowledge needed to begin to explore properties of circles?

Suggested materials:

- unlined paper
- compass
- straightedge
- construction paper and/or other materials for creating a circle book

Warm-up: Post the following.

Use a compass to construct a circle on an unlined sheet of paper. Label the center of your circle.

a. What information do you need to determine a unique circle?

b. Use your answer to 'Item a' to help you write a definition of a circle.

Opening: Discuss the warm-up. Students worked with circles in middle grades and should know the definition of a circle, radius, diameter, and circumference. They should also know how to find the circumference and area of a circle. The warm-up (problem 1 of the task) is intended to remind students of the definition of a circle.

We are recommending that students keep a *Circle Book* for this unit. The book should be the student's own reference for the definitions and theorems learned in this unit. You might begin by discussing your expectations for this reference book.

One method of addressing the terminology and notation introduced in this task is to have students do a *pair read* of the information between problems 1 and 2 of the task. Have students work in pairs. One student reads a small "chunk" of the material and the other student summarizes and /or clarifies what was read. Students develop a common understanding of the information. Students then swap roles to read and summarize the next bit of information.

After the reading has been accomplished, assign terms to various students. Have those students present the material to the class, including illustrations that may be placed on the board or on the word wall.

Worktime: Students should complete problems 2-4 of the task.

After students have had time to complete problem 3, have a brief, whole-class discussion of problems 2 and 3 to be sure that all students understand critical terms.

Problem 4 requires that students prove two theorems. The proofs of these theorems rest on the triangle congruence properties and give students excellent opportunities to begin to see circle properties as applications of triangle congruence and similarity. As you monitor student work, look for and encourage students who are writing formal two column proofs as well as those who are writing paragraph proofs. Both should be shared during the closing.

Problem 4 includes the first of several times that students will be asked to write a theorem and its converse as a biconditional statement.

Closing: Discuss problem 4. Both two column and paragraph proofs should be shared. Statements, their converses, and biconditional statements should be discussed thoroughly.

Homework: Homework is included after the student task. We strongly recommend that these problems be assigned. Students may need practice in addition to the problems provided here.

Differentiated support/enrichment:

Check for Understanding: Use your compass and straightedge to create a circle containing each of the following. In each case, use the appropriate notation.

- a central angle
- a major arc
- a minor arc

- a semicircle
- a chord

Resources/materials for Math Support: Students should preview;

- definitions of circle, radius, diameter, circumference
- degree measure of a complete circle is 360°
- converse of a statement
- biconditional statements
- proving triangles congruent

Mathematics II

Relationships among Central Angles, Arcs, and Chords

Day 1 Student Task

In this unit you will study properties of circles. As you progress through the unit, you will learn new definitions and develop and prove numerous theorems. It will be advantageous for you to keep a "**Circle Book**" that includes the definitions and theorems addressed in each task. With each definition and theorem you enter, you should also include an illustrative sketch.

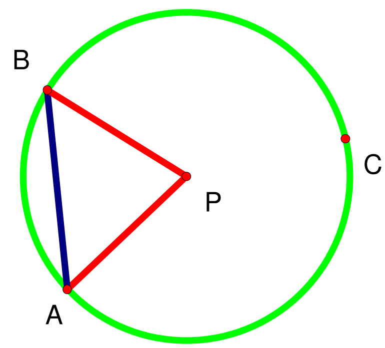
We will begin by re-visiting the definition of a circle.

1. Use a compass to construct a circle on an unlined sheet of paper. Label the center of your circle.
   a. What information do you need to determine a unique circle?

   b. Use your answer to *Item a* to help you write a definition of a circle.

Now we will introduce some notation and terminology needed to study circles.
Consider the figure at right.

$m\angle APB = 75^0$



Circles are identified by the notation $\odot P$, where P represents the point that is the center of the circle.

A **central angle** of a circle is an angle whose vertex is at the center of the circle. $\angle APB$ is a central angle of $\odot P$.

A portion of a circle's circumference is called an **arc**. An arc is defined by two endpoints and the points on the circle between those two endpoints. If a circle is divided into two unequal arcs, the shorter arc is called the **minor arc** and the longer arc is called the **major arc**. If a circle is divided into two equal arcs, each arc is called a **semicircle**.

In our figure, we call the portion of the circle between and including points A and B, arc AB notated by $\overset{\frown}{AB}$. We call the remaining portion of the circle arc ACB, or $\overset{\frown}{ACB}$. Note that major arcs are usually named using three letters.

We say that the central angle $\angle APB$ *intercepts* or has $\overset{\frown}{AB}$. We also say that $\overset{\frown}{AB}$ *subtends* or has the central angle $\angle APB$. Note that when we refer to the arc of a central angle, we usually mean the minor arc unless otherwise stated.

Arcs are measured in two different ways - using degree measure and using linear measure. Usually when we refer to the **measure** of an arc, we are referring to the degree measure. The **measure** of a minor arc is defined to be the measure of the central angle that intercepts the arc. The measure of a major arc is 360° minus the measure of the minor arc with the same endpoints. In the figure above, the measure of $\overset{\frown}{AB}$ is 75° because that is the measure of its central angle. The measure of $\overset{\frown}{ACB}$ is 360° – 75° or 285°.

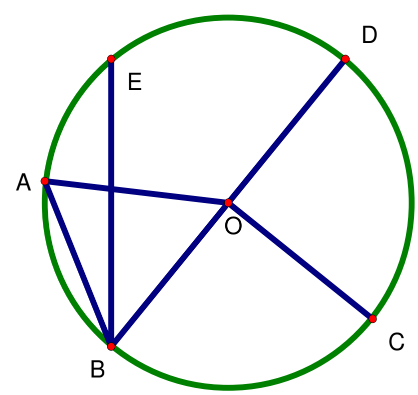
The **length** of an arc is different from its measure. The length is given in linear units and is determined as a portion of the length of the entire circumference of the circle. We will investigate the length of an arc in a later task. **Congruent arcs** have equal degree measures and equal lengths.

A **chord** is a *segment* whose endpoints lie on the circle. In the above figure, segment $\overline{AB}$ is a chord of $\odot P$.

2. How many chords can be in a circle?

   What is the longest chord in a circle? Explain how you know?

3. Refer to the figure at the right. Identify and name each of the following. Be sure to use the correct notation.



   a. Two different central angles
   b. A minor arc
   c. A major arc
   d. A semicircle
   e. Two different chords
   f. The central angle subtended by $\overset{\frown}{AD}$

   Use your protractor to help you find the following measures:

   g. The measure of $\overset{\frown}{AC}$
   h. The measure of $\overset{\frown}{DEC}$

4. Now it is time to use some of the terminology you have learned. Consider the following two theorems:

   In the same circle or congruent circles, if two chords are congruent, their intercepted arcs are congruent.

In the same circle or congruent circles, if two arcs are congruent, then their chords are congruent.

a. Prove that each of the theorems is true.

b. Write the two theorems as one biconditional statement.

Mathematics II
Relationships among Central Angles, Arcs, and Chords
Day 1 Homework

1. In $\odot P$, radii $\overline{PA}$, $\overline{PB}$, and chord $\overline{AB}$ are drawn. $PA = 2x + 3$, $PB = 3x - 7$, and

$AB = 43 - 2x$.

a. Draw a sketch to illustrate the given information.

b. Find the measure of $\overset{\frown}{AB}$. Show how you know.

2. In $\odot P$, radii $\overline{PA}$ and $\overline{PB}$ are drawn. $PA = \frac{3}{5}x - 2$ and $PB = x - 10$.
   a. Draw a sketch to illustrate the given information.

   b. Find the length of a diameter of the circle.

3. In $\odot P$, radii $\overline{PA}$ and $\overline{PB}$ are drawn. If radius $\overline{PC}$ bisects $\angle APB$, prove that $\overline{AC} \cong \overline{BC}$.

Mathematics II
Task 1: ***Relationships among Central Angles, Arcs, and Chords*** **Day 2/2**
(GaDOE TE Task 1: # 4)

MM2G3. Students will understand the properties of circles.

a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.
b. Understand and use properties of central, inscribed, and related angles.
d. Justify measurements and relationships in circles using geometric and algebraic properties.

New vocabulary:

Mathematical concepts/skills:

- distance from the center of a circle to a chord of the circle
- constructing the perpendicular to a segment through a given point not on the segment
- converse of a given statement
- biconditional statements
- proof or justification of the following statements:
    - *Two chords of the same circle are congruent if and only if they are equidistant from the center of the circle.*
    - *A radius of a circle is perpendicular to a chord if and only if it bisects the chord.*
    - *A radius of a circle perpendicular to a chord bisects the arc intercepted by the chord.*
    - *Any line that is a perpendicular bisector of a chord must contain the center of the circle.*

Prior knowledge:

- construction of a perpendicular line to a segment through a given point
- converse of a statement
- biconditional statement
- triangle congruence postulates

Essential question(s): What definitions and theorems do I already know that will allow me to develop and prove theorems related to chords of a circle?

Suggested materials:

- unlined paper
- compass
- straightedge

Warm-up: Have students compare homework with a partner. Tell them to be prepared to ask questions related to any problems they still do not understand.

Opening: Discuss the homework as a means of reviewing material from the previous lesson and setting the stage for the properties of chords to be studied in today's lesson.

Worktime: Students should complete problems 5 – 7 of the task. Proofs required in problems 5 and 6 are fairly simple and will give students more opportunities to use triangle congruence postulates learned in Math I. Note that in problems 6c and 7, students are asked to justify their thinking as opposed to developing formal proofs.

Problem 8 asks students to apply the information learned in problem 7. Hopefully students will illustrate their thinking by using the compass to draw a piece of a circle; then draw at least two chords and construct perpendicular bisectors to the chords. Rulers can be used to measure the distance from the circle to the point of intersection of the two perpendicular bisectors.

Closing: Problems 5 – 8 should be discussed thoroughly. Biconditional statements that result in theorems should be discussed and agreed upon as a class. (See the theorems listed in the *mathematical concepts and skills* section above.)

Problem 8 may be assigned as homework if time does not permit that it be completed in class. This is an extremely important application, however, and should be discussed.

Homework: In addition to problem 8, we recommend traditional skills problems related to the theorems developed in problems 5 – 7.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview;

- construction of a perpendicular line to a line through a point not on the line
- construction of the perpendicular bisector of a segment
- converse of a statement
- biconditional statements
- proving triangles congruent

Activities labeled as *Perpendicular Chord Bisector* and *Circle Center*, which can be found at the Geogebra site http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29, contain discovery and skills practice related to the concepts in this task. Other materials can be found on the GaDOE 's *Learning Village*.

Mathematics II
Relationships among Central Angles, Arcs, and Chords
Day 2 Student Task

5. Use a compass to construct a circle on an unlined sheet of paper. Label the center of your circle.

   a. Draw any chord, other than a diameter, on your circle. Use your compass and a straightedge to construct a segment that represents the distance from the center of your circle to the chord. What is the relationship between the chord and the segment representing this distance?

   b. Mary made the following conjecture: If two chords of a circle are the same distance from the center of the circle, the chords are congruent. Mary is correct. Use what you learned in *Item 5a* to help Mary prove her conjecture.

   c. State the converse of Mary's conjecture and prove that it is true.

   d. Write Mary's conjecture and its converse as a biconditional statement.

   e. When a conjecture has been proven, it can be stated as a theorem. Write and illustrate this theorem in your *Circle Book.*

6. Ralph made the following conjecture: A radius perpendicular to a chord bisects the chord.

   a. Use your construction from *Item 5a* to help you prove that Ralph's conjecture and the converse are true.

   b. Write Ralph's conjecture and its converse as a biconditional statement and illustrate it in your *Circle Book.*

   c. Ralph also believes that a radius perpendicular to a chord bisects the arc intercepted by the chord. Is this true? How do you know?

7. Tevante examined his construction and his partner's construction. He believes that *any* line that is a perpendicular bisector of a chord of a circle must also contain the center of the circle. Is he right? How do you know?

8. An investigator working for the Georgia Bureau of Investigation's crime lab has uncovered a jagged piece of a circular glass plate believed to have been used as a murder weapon. She needs to know the diameter of the plate. How might you use the information you learned in problem 7 to help determine the diameter of the circular plate?

   Use a compass, a straightedge, and a ruler to illustrate your answer.

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Mathematics II: Unit 3
Circles and Spheres

Task 2: Investigating Circle Relationships

Mathematics II
Task 2: ***Investigating Circle Relationships*** **Day 1/1**

MM2G3. Students will understand the properties of circles.
a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.
b. Understand and use properties of central, inscribed, and related angles.
d. Justify measurements and relationships in circles using geometric and algebraic properties.

New vocabulary: inscribed angle, inscribed polygon

Mathematical concepts/skills:
- measuring angles using a protractor
- development and proof of the following theorems:
  - The measure of an angle inscribed in a circle is one-half the measure of its intercepted arc.
  - Opposite angles of a quadrilateral inscribed in a circle are supplementary.
  - The measure of an angle formed by two intersecting chords is one-half the sum of the measures of the two intercepted arcs.

Prior knowledge:
- measuring angles using a protractor
- exterior angle theorem
- triangle congruence postulates
- sum of the measures of the angles of a quadrilateral

Essential question(s): How can I determine the measure of an angle inscribed in a circle? How can knowing the measures of angles inscribed in a circle help me develop other circle properties?

Suggested materials:
- unlined paper
- compass
- straightedge
- protractor

Warm-up: Ask students to read the introduction to the task and to complete problems 1 and 2.

Opening: Discuss the fact that students will be conducting three fairly short investigations in this task beginning with the relationship between an inscribed angle and its intercepted arc.

Ask a student to describe an inscribed angle. Allow students to share the constructions they have created. Be sure to have students name and use the correct notation for their inscribed angles and for the intercepted arcs.

Worktime: Students should work in groups of 2 or 3.

Investigation 1: Students will first determine that the measure of an inscribed angle is equal to one-half the measure of the central angle that intercepts the same arc. Once students have had a chance to gather data from classmates, you may want to compile the class data comparing the measures of the inscribed angles and the corresponding central angles. A whole-class discussion of measurement error may be needed.

It is important to remind students that the conjecture they are asked to make in problem 5 is about the relationship between the inscribed angle and its *intercepted arc*. If necessary, ask guiding questions that will help students use the information learned in the previous task to link the measure of the central angle to the measure of the intercepted arc.

Allow several students to share their conjectures before discussing the proof. Students should make conjectures in their own words but statements may need to be honed. After class discussion, student conjectures should be in a form very close to the following statement:

> *The measure of an angle inscribed in a circle is equal to one-half the measure of its intercepted arc.*

Note that students should prove case 1 of this theorem and then use it to prove the remaining two cases. The proof is based on the fact that $\angle APB$ is an exterior angle of the isosceles triangle APC. Individual teachers should decide how much scaffolding their students need in developing this proof. However, all students should be exposed to the complete proof.

Investigation 2: If a discussion of measurement error was had during *Investigation 1*, this task should move quickly. Note that students are asked to justify their conjecture here as opposed to writing a formal proof. Student conjectures should be a form of the following statement:

> *Opposite angles of a quadrilateral inscribed in a circle are supplementary.*

Investigation 3: Monitor student work on problem 3 very carefully. Many students will make the false assumption that the measure of $\overset{\frown}{AB}$ is equal to the measure $\angle AEB$. However, $\angle AEB$ is not a central angle. Students must draw and measure central angles in order to get the measures of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$.

The conjecture required in this investigation should be some form of the following statement:

> *The measure of an angle formed by two chords intersecting within a circle is equal to one-half the sum of the measures of the intercepted arcs.*

Students have been provided a hint to help them begin the proof. Considering that $\overline{BD}$ has been drawn in the diagram, the proof is based on the fact that $\angle AEB$ is an exterior angle of $\triangle BED$. The measure of inscribed $\angle EBD$ is one-half the measure of $\overset{\frown}{CD}$. Likewise the measure of inscribed $\angle BDE$ is one-half the measure of $\overset{\frown}{AB}$. The proof follows naturally from this information.

Closing: Whole-class discussions should be held at various intervals of each investigation. (See suggestions in the *worktime*.) Class discussions of each conjecture should be held before respective proofs are begun. Students should include each theorem, along with an illustration, in their *Circle Books*.

Homework: Traditional skills problems related to the theorems developed in this task are recommended.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview;

- measuring angles using a protractor
- the exterior angle theorem
- proof using the triangle congruence postulates
- sum of the measures of the angles of a quadrilateral

Discovery and skills practice related to the concepts in this task can be found at the Geogebra site: http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29 and also on the GaDOE 's *Learning Village*.

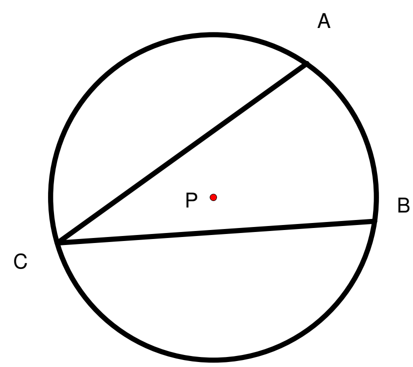
Investigating Circle Relationships
Day 1 Student Task

In this task you will investigate three different relationships:

- the relationship between an inscribed angle and its intercepted arc,
- relationships among angles of a quadrilateral inscribed in a circle, and
- the relationships between measures of angles formed by intersecting chords and their intercepted arcs.

Investigation 1: Examining Inscribed Angles

In a circle, an **inscribed angle** is an angle whose vertex lies on the circle and whose sides are chords of the circle. In $\odot P$ below, $\angle ACB$ is an inscribed angle.



We want to determine the relationship between the measures of inscribed angles and their intercepted arcs.

1. Begin by using your compass to construct a circle. Label the center of your circle.

2. Use your straightedge to construct an inscribed angle. Label the vertex of your inscribed angle and the points where the sides of the angle intersect the circle. What is the name of your inscribed angle? What is the name of the arc your angle intercepts?

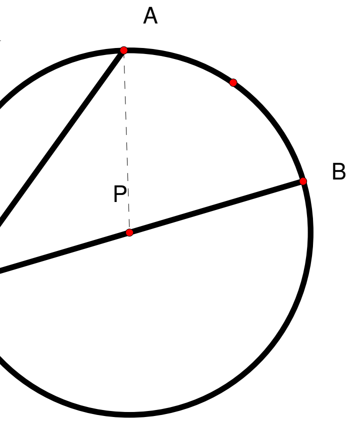
3. Use your straightedge to construct the central angle that intercepts the same arc intercepted by your inscribed angle.

4. Now use your protractor to measure both your inscribed angle and the central angle that you have drawn. Can you determine any relationship between these two measures? What do these measures tell you about the relationship between the measure of the inscribed angle and its intercepted arc?

5. Collect measures of inscribed angles and the corresponding central angles from at least 5 of your classmates. Based on the information that you have so far, make a conjecture about the relationship between the measure of an inscribed angle and its intercepted arc?

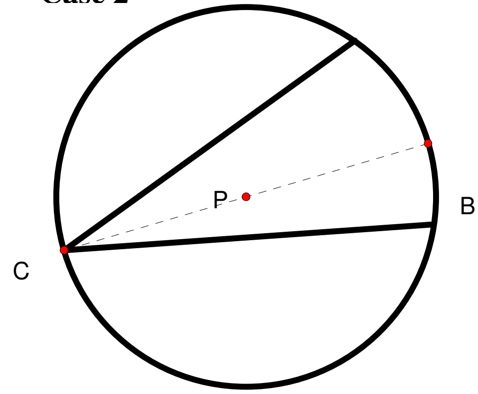
6. Remember that a conjecture is not a theorem until it has been proved. There are three different situations (cases) that can arise when considering inscribed angles. In the diagrams below, each case is explained. The dashed segment added to each picture is a hint that should help you prove your conjecture. Prove your conjecture by proving case 1 first. Then use case 1 to prove cases 2 and 3.

Case 1

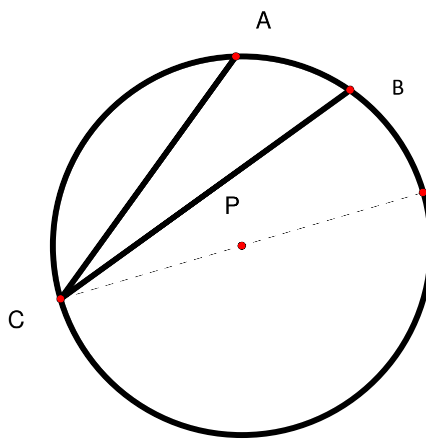


The center of the circle lies on one side of the inscribed angle.

Case 2



The center of the circle is in the interior of the inscribed angle.



Case 3

The center of the circle lies in the exterior of the angle.

7. The theorem you have just proved will be used to help you prove other theorems and to solve problems related to circles. Be sure to include it in your *Circle Book*.

Investigation 2: Determining Relationships among Angles of a Quadrilateral Inscribed in a Circle

A polygon is **inscribed** in a circle when every vertex of the polygon is on the circle. In the next sequence of items, we will investigate the relationships among the angles of a quadrilateral inscribed in a circle.

1. Begin by using your compass to construct a circle. Label the center of your circle.

2. Choose four points on your circle and then use your straightedge to construct an inscribed quadrilateral. Label the vertices of your quadrilateral.

3. Now use your protractor to measure all four angles of the quadrilateral. Can you make any determination about the sum of the measures of all four angles? How does this compare with what you already know about the measures of the angles of a quadrilateral?

4. Can you make any determination about the measures of the *opposite* angles of your inscribed quadrilateral?

5. Collect measures of opposite angles of inscribed quadrilaterals from at least 5 of your classmates. Based on the information that you have so far, make a conjecture about the relationship between the measures of the opposite angles of an inscribed quadrilateral.

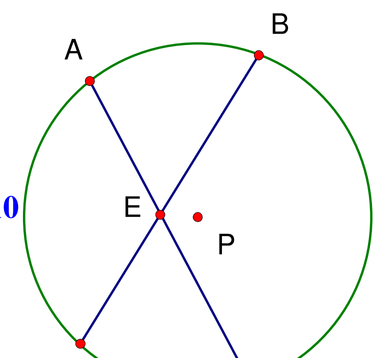
6. Use what you learned about inscribed angles in the first part of this task to help you prove your conjecture. After your conjecture has been proved, add it, along with an illustration, to the other theorems in your *Circle Book*.

Investigation 3: Examining Angles Formed by Two Chords Intersecting within a Circle

In this investigation we will determine the relationship between measures of angles formed by intersecting chords and the measures of the arcs intercepted by those angles.

1. Use your compass to construct a circle. Label the center of your circle.

2. Construct two chords that intersect within the circle. Label endpoints of your chords *A*, *B*, *C*, and *D* and the point of intersection of the chords *E* in the manner shown below. (The diagram given is for labeling purposes only. Your chords may be anywhere in your circle as long as they intersect within the circle.)

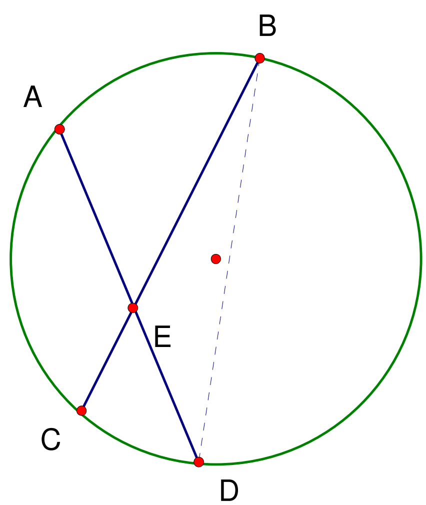


3. Use your construction and your protractor to investigate the relationship between the measure of $\angle AEB$ and the measures of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$.

4. Compare findings with at least 5 other classmates. You might want to put your findings and the findings of your classmates in a table.

5. Make a conjecture related to the measure of an angle formed by intersecting chords and the measures of the intercepted arcs?

6. Work with members of your group to prove your conjecture. The figure below provides you with a hint.



ATLANTA PUBLIC SCHOOLS
Mathematics & Science Initiative
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Teacher's Curriculum Supplement

Mathematics II: Unit 3
Circles and Spheres

Task 3: Sunrise on the First Day of the New Year

Mathematics II
Task 3: ***Sunrise on the First Day of the New Year*** **Day 1/1**

MM2G3. Students will understand the properties of circles.
a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.

b. Understand and use properties of central, inscribed, and related angles.
d. Justify measurements and relationships in circles using geometric and algebraic properties.

New vocabulary: tangent line, secant line

Mathematical concepts/skills:

- definitions of tangent line and secant line
- relationships between the distance from the center of a circle to a line and the length of a radius of the circle
- perpendicularity of a tangent to a radius of a circle at the point of tangency
- use of dynamic geometry software to investigate properties of circles
- development and proof of the following theorems:
    - The measure of an angle formed by two secants that intersect outside a circle is one-half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.
    - The measure of an angle formed by a secant and a tangent that intersect outside a circle is one- half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.
    - The measure of an angle formed by the intersection of a tangent and a chord at the point of tangency is one-half the measure of the intercepted arc.
    - The measure of an angle formed by two tangents that intersect outside a circle is one-half the difference of the larger intercepted arc measure and the smaller intercepted arc measure.

Prior knowledge:

- measuring angles using a protractor
- exterior angle theorem

Essential question(s): How can I determine the measures of angles created by tangents, secants, and chords drawn to a circle? How can I use these measures to solve problems related to circles?

Suggested materials:

- unlined paper
- compass
- straightedge
- protractor
- dynamic geometry software

Warm-up: Ask students to read the introduction to the task and to complete problems 1 and 2.

Opening: Discuss problems 1 and 2.

Worktime: Allow students to work in pairs to complete problems 3 – 6. After students have had ample time to complete these problems, have a short, whole-class discussion addressing all four items.

The remainder of the task includes four investigations. We suggest that students work in groups of three or four and that each group be assigned one investigation. Upon completion of the investigation, each group should prepare a presentation to be shared with the class. Each presentation should include the following:

- a diagram of the situation being investigated,
- demonstration or explanation of the manner in which the investigation was conducted,
- the data gathered,
- a complete statement of the conjecture formed, and
- a complete proof of the conjecture in either two-column or paragraph form.

It will be necessary for multiple groups of students to perform the same investigation. Teachers may choose to ask groups performing the same investigation to use different tools (i.e., dynamic software; or compass, straightedge and protractor).

Closing: Each investigation should be presented to the class. Teachers may want to randomly choose one group to make the formal presentation of an investigation and then allow other groups having performed the same investigation to share any additional insights or make corrections.

All students should include all four theorems discovered in this task in their *Circle Books*, along with diagrams.

Homework: See homework following the Student Task.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview;

- measuring angles using a protractor
- the exterior angle theorem

Activities labeled as *Two Secants* and *Two secants Practice*, which can be found at the Geogebra site http://www.geogebra.org/en/wiki/index.php/Circles_%28Angles%29, contain discovery and skills practice related to the concepts in task. Other materials can be found on the GaDOE 's *Learning Village*.

Mathematics II
Sunrise on the First Day of the New Year
Day 1 Student Task

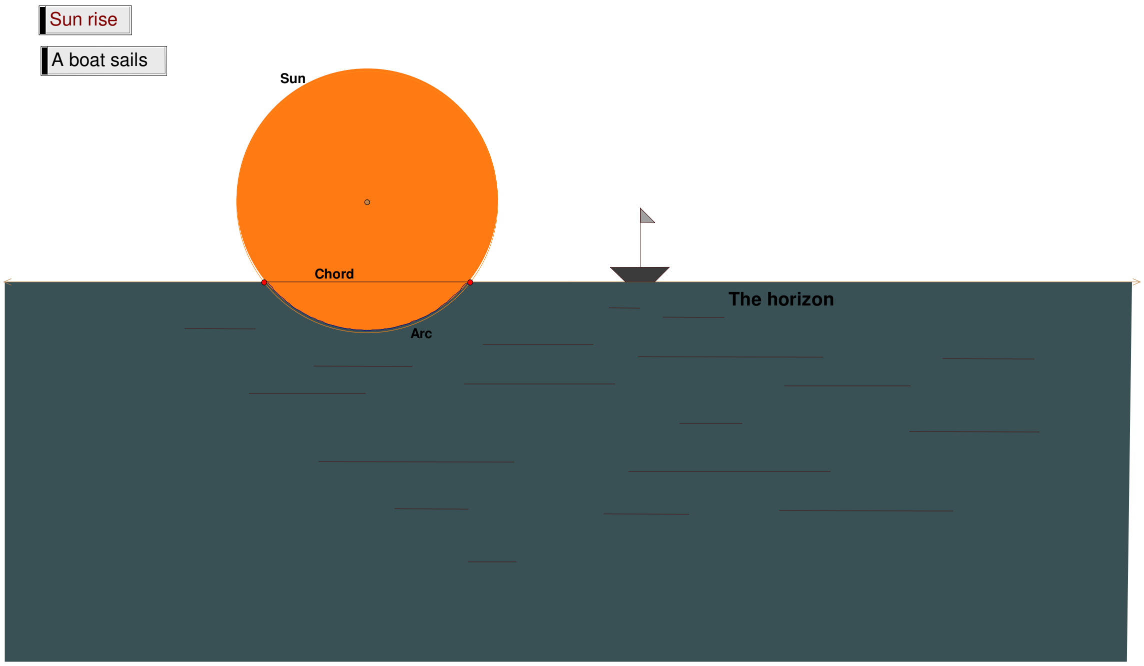
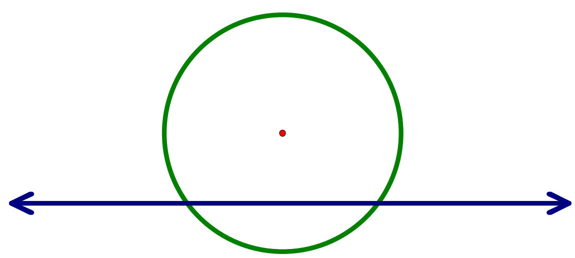


Figure 1: Sunrise

It is customary for people in Asia to visit the seashores on the eastern sides of their countries on the first day of the year. While watching the sun rise over the ocean, visitors wish for good luck in the New Year.

As the sun rises, the horizon cuts the sun at different positions. Although a circle is not a perfect representation of the sun, we can simplify this scene by using a circle to represent the sun and a line to represent the horizon.



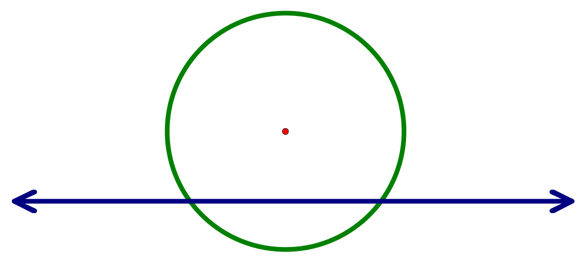
1. Why is a circle not a perfect representation of the sun? Why is our use of a circle and a line still an appropriate representation of the sunrise?

2. Using the simplified diagram above, sketch and describe the different types of intersections the sun and the horizon may have.

3. A **tangent line** is a line that intersects a circle in exactly one point. A **secant line** intersects a circle in two points. Do any of your sketches in *item 2* contain tangent or secant lines? If so, label them. Is it possible for a line to intersect a circle in 3 points? 4 points? Explain why or why not.

4. When a secant line intersects a circle in two points, it creates a chord. As you have already learned, a **chord** is a segment whose endpoints lie on the circle. How does a chord differ from a secant line?

5. Look again at our representation of the sun and the horizon.



Let d represent the distance between the center of a circle and a line l. Let r represent the length of a radius of the circle. Describe the relationship between d and r when l:

   a. is a tangent line

   b. is a secant line

   c. does not intersect the circle

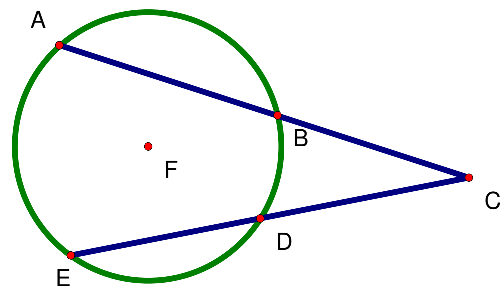
6. In *Item 5a* you compared the length of a radius of a circle to the distance from the center of the circle to a tangent line. What does this comparison tell you about the relationship of a tangent line to a radius at the point of tangency. Explain your thinking.

In the remaining items of this task, we will investigate the relationships between the measures of angles formed by secants, tangents, and chords to the measures of the arcs intercepted by these angles. We will explore four different situations:

- angles formed by two secants intersecting at a point outside a circle,
- angles formed by a secant and a tangent that intersect outside a circle,
- angles formed by a tangent and a chord of the circle, and
- angles formed by two tangents that intersect outside a circle.

Investigation 1: In this investigation we will explore the relationship between the measure of an angle formed by two secants that intersect outside a circle and the measures of its intercepted arcs.

1. In the diagram below, $\angle ACE$ is formed by secant lines that intersect at point *C outside* circle *F*. Why do we say this angle is formed by secant lines?

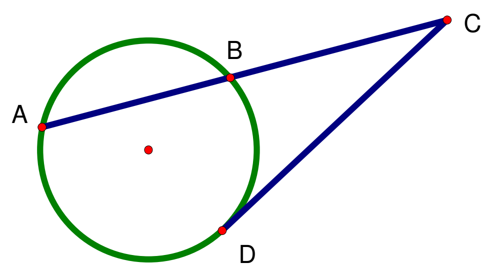


2. Use the tools or software program provided by your teacher to construct a diagram like the one show above. Investigate the relationship between the measure of $\angle ECA$ and the measures of $\overset{\frown}{BD}$ and $\overset{\frown}{AE}$. Examine several sets of measures for the indicated angle and arcs by checking with at least three other classmates or by using dynamic software.

3. Make a conjecture about the relationship between the measure of $\angle ECA$ and its intercepted arcs?

4. Prove that your conjecture is always true. (Hint: In the figure above, draw $\overline{EB}$ and begin by considering the relationship among $\angle ABE$, $\angle BED$, and $\angle ECA$.) Once you have proved the conjecture, it becomes a theorem. Add this theorem and a diagram to your *Circle Book*.

Investigation 2: In the diagram below, $\angle ACD$ is formed by a secant line and a tangent line that intersect at the point *C outside* $\odot F$. We want to explore the relationship between this angle and its intercepted arcs.

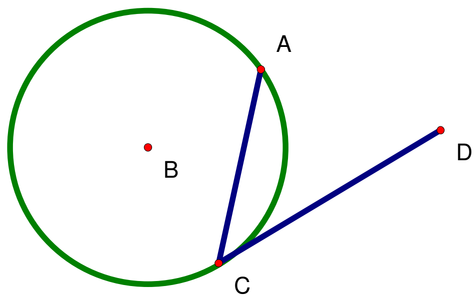


1. Use the tools or software program provided by your teacher to construct a diagram like the one shown above. Investigate the relationship between the measure of $\angle DCA$ and the measures of $\overset{\frown}{BD}$ and $\overset{\frown}{AD}$. Examine several different sets of measures for the indicated angle and arcs by checking with at least three other classmates or by using dynamic software.

2. Make a conjecture about the relationship between the measure of $\angle DCA$ and its intercepted arcs?

3. Prove that your conjecture is always true. (Hint: In the figure above, draw $\overline{DB}$ and develop a proof similar to the one used in *Item 4* of *Investigation 1*. Once you have proved the conjecture, it becomes a theorem. Add this theorem and a diagram to your *Circle Book*.

Investigation 3: Next we consider the relationship between the measure of an angle formed by a tangent and a chord at the point of contact and its intercepted arc.

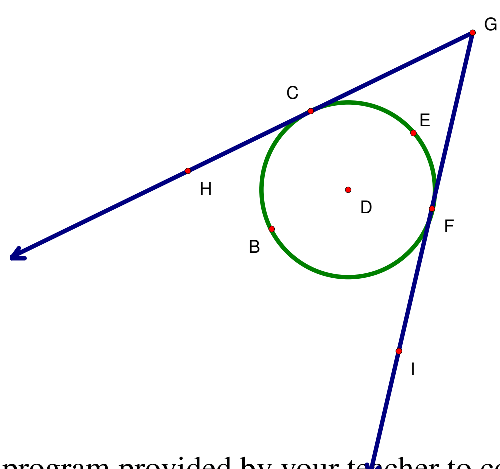


1. Use the tools or software program provided by your teacher to construct a diagram like the one show above. Investigate the relationship between the measure of $\angle DCA$ and the measure of $\overset{\frown}{AC}$. Examine several different sets of measures for the indicated angle and arc by checking with at least three other classmates or by using dynamic software.

2. Make a conjecture about the relationship between the measure of $\angle DCA$ and its intercepted arc?

3. Prove that your conjecture is always true and add this theorem to your *Circle Book.*

Investigation 4: Finally, we investigate the relationship between the measure of an angle formed by two tangents that intersect in a point outside a circle and its intercepted arcs.



1. Use the tools or software program provided by your teacher to construct a diagram like the one show above. Investigate the relationship between the measure of $\angle FGC$ and the measures of $\overset{\frown}{CF}$ and $\overset{\frown}{CBF}$. Examine several different sets of measures for the indicated angle and arcs by checking with at least three other classmates or by using dynamic software.

2. Make a conjecture about the relationship between the measure of $\angle FGC$ and its intercepted arcs?

3. Prove that your conjecture is always true and add this theorem to your *Circle Book*. (Hint: Construct $\overline{CF}$ and consider the relationship among $\angle FCH$, $\angle CFG$, and $\angle FGC$).

Mathematics II
Sunrise on the First Day of the New Year
Day 1 Homework

1. The angle formed by two tangents drawn to a circle from the same external point measures 72°. Find the measure of the smaller of the intercepted arcs.

2. One of the arcs intercepted by two tangents drawn to a circle from an external point measures 130°. Find the measure of the angle formed by the tangents.

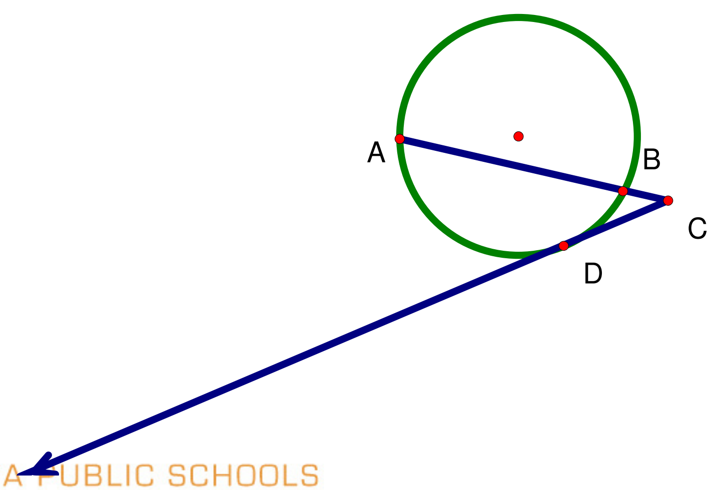
3. Find the measure of an angle formed by a tangent and a secant drawn to a circle from an external point if they intercept arcs whose measures are:
   a. 140° and 35°
   b. x and y

4. From a point P outside a circle, two secants $\overline{PAB}$ and $\overline{PCD}$ are drawn. $\angle P$ contains 43° and $\overset{\frown}{AC}$ contains 80°. Find the measure of $\overset{\frown}{CD}$.

5. Find the measure of an arc intercepted by an angle formed by a tangent and a chord drawn from the point of contact if the angle measures 23°.

6. Equilateral $\triangle ABC$ is inscribed in a circle. Find the measure of the acute angle formed by side $\overline{BC}$ and the tangent at C.

7. In the figure below, $\overline{CBA}$ is a secant drawn to the circle and $\overline{CD}$ is a tangent. The measure of $\angle C$ is represented by x, and the measures of $\overset{\frown}{BD}$, $\overset{\frown}{DA}$, and $\overset{\frown}{AB}$ are represented by y, $4x - 10$, and $8x$, respectively. Find the number of degrees contained in $\overset{\frown}{ABD}$. (Hint: This may require that you solve a system of linear equations.)



Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 3
Circles and Spheres

Task 4: Investigating Lengths of Segments

Mathematics II
Task 4: ***Investigating Lengths of Segments*** **Day 1/1**

MM2G3. Students will understand the properties of circles.
a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.
b. Understand and use properties of central, inscribed, and related angles.

d. Justify measurements and relationships in circles using geometric and algebraic properties.

New vocabulary:

Mathematical concepts/skills:

- use of dynamic geometry software to investigate relationships among lengths of segments formed by chords, secants, and tangents,
- verification and proof of the following theorems:
  - If two chords intersect in a circle, the product of the lengths of the segments of one chord equals the product of the segments of the other chord.
  - If two secant segments are drawn to a circle from the same external point, the product of the length of one secant segment and its external part is equal to the product of the length of the other secant segment and its external part.
  - If a secant segment and tangent segment are drawn to a circle from the same external point, the product of the length of the secant segment and its external part equals the square of the length of the tangent segment.
  - Tangent segments drawn to a circle from a point outside the circle are congruent.

Prior knowledge:

- proving triangles similar
- ratios of lengths of corresponding sides of similar triangles are equal
- proving triangles congruent

Essential question(s): What relationships exist among the lengths of segments created by tangents, secants, and chords drawn to a circle? How can I use these relationships to solve problems related to circles?

Suggested materials:

- unlined paper
- compass
- straightedge
- ruler
- protractor
- dynamic geometry software

Warm-up: Have students compare homework from the previous lesson with a partner. Tell them to be prepared to ask questions related to any problems they still do not understand.

Opening: Discuss the homework from Task 3 as a means of reviewing and setting the stage for the current lesson.

Worktime: Students should work in groups of 3 or 4. We suggest that teachers allow each group of students to choose 3 of the 4 investigations to complete, making sure that each investigation is completed by at least one group of students. Note that in this task, students are *given* the relationships to be examined in all but the last investigation.

Upon completion of their investigations, each group should prepare a presentation to be shared with the class. Each presentation should include the following:

- a diagram of the situation being investigated,
- demonstration or explanation of the manner in which the investigation was conducted,
- the data gathered,
- a complete statement of the conjecture formed, and
- a complete proof of the conjecture in either two-column or paragraph form.

Teachers may choose to ask groups performing the same investigations to use different tools (i.e., dynamic software; or compass, straightedge, and rulers).

Closing: Each investigation should be presented to the class. Teachers may want to randomly choose one group to make the formal presentation of an investigation and then allow other groups having performed the same investigation to share any additional insights or make corrections.

All students should include all four theorems discovered in this task in their *Circle Books*, along with diagrams.

Homework: See homework following the Student Task.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview;

- proving triangles similar
- ratios of lengths of corresponding sides of similar triangles are equal
- proving triangles congruent

The following geogebra websites contain discovery and skills practice related to the concepts in this task.

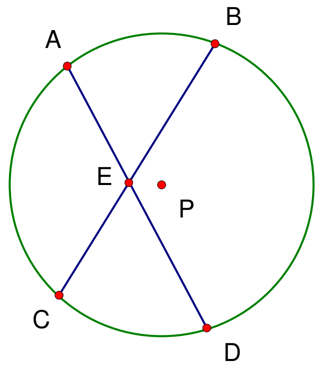
http://www.geogebra.org/en/upload/files/UC_MAT/chords_in_a_circle.html
http://www.geogebra.org/en/upload/files/UC_MAT/chords_outside_a_circle.html

Mathematics II
Investigating Lengths of Segments
Day 1 Student Task

In the previous task we investigated the measures of angles formed by secants, tangents, and chords. In this task we will examine relationships among segment lengths when the segments are created by chords, secants, and tangents of a circle.

Investigation 1: Lengths of Segments Formed by Two Intersecting Chords

In this investigation, you will examine lengths of segments formed by two intersecting chords.



In the figure shown here, the intersecting $\overline{AD}$ and $\overline{BC}$ form four segments: $\overline{AE}$, $\overline{ED}$, $\overline{BE}$, and $\overline{EC}$. The following relationship exists among the lengths of these four segments:

$$AE \cdot ED = BE \cdot EC$$

1. Use the tools or software provided by your teacher to construct a figure like the one above. Then verify the given relationship using your own construction.

2. Obtain several different sets of measures for the four segments being investigated by comparing information with at least 3 other classmates or by using dynamic software. Does the relationship hold for these measures?

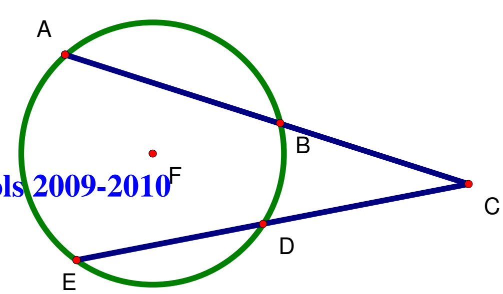
3. State the relationship given above as a conjecture.

4. Use your construction or the picture above to help you prove that this conjecture is always true for segments formed by intersecting chords. (Hint: The proof can be based on similar triangles.)

5. Once you have proven the conjecture, don't forget to include it, along with a diagram, in your *Circle Book.*

Investigation 2: Lengths of Segments Formed by Secants Intersecting at a Point Outside a Circle

In the diagram below, segments $\overline{AC}$ and $\overline{EC}$ are formed by secant lines that intersect at point C *outside* $\odot F$.



We will investigate the following relationship:

$$CD \cdot CE = CB \cdot CA$$

1. Use the tools or software provided by your teacher to construct a figure like the one above. Then verify the given relationship using your own construction.

2. Obtain several different sets of measures for the four segments being investigated by comparing information with at least 3 other classmates or by using dynamic software. Does the relationship hold for these measures?

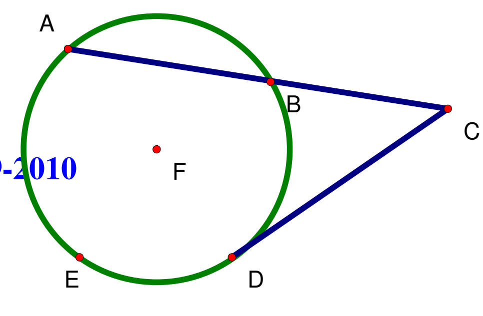
3. The conjecture you are investigating can be stated as follows:

   If two secant segments are drawn to a circle from the same external point, the product of the length of one secant segment and its external part is equal to the product of the length of the other secant segment and its external part.

   Prove that the conjecture is always true. Once you have proved the conjecture, it becomes a theorem. Add this theorem and a diagram to your *Circle Book*.

Investigation 3: Lengths of Segments Formed by a Tangent and a Secant Intersecting at a Point Outside a Circle

Now we investigate segment lengths of a tangent and a secant that intersect in a point outside a circle. A diagram is shown here.



1. Which segment is formed by a tangent? How do you know? What is the longest segment formed by a secant?

2. We want to investigate the following relationship:

   $CD^2 = CB \cdot CA$

   Use the *Geogebra* website given below (or other dynamic geometry software) to help you examine this relationship.

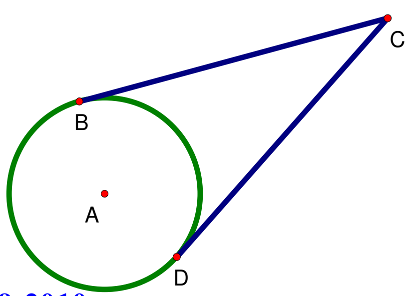
   http://www.geogebra.org/en/upload/files/UC_MAT/chords_outside_a_circle.html

3. The relationship you are investigating can be stated as follows:

   If a secant segment and tangent segment are drawn to a circle from the same external point, the product of the length of the secant segment and its external part equals the square of the length of the tangent segment.

   Prove that this relationship is always true.

Investigation 4: Lengths of Segments Formed by Two Tangents Intersecting Outside a Circle



In the diagram shown here, $\overline{BC}$ and $\overline{DC}$ are formed by two segments tangent to $\odot A$ at B and D respectively. The tangents are drawn from point C outside the circle.

1. Use the tools or software provided by your teacher to construct a figure like the one above. Use your figure to determine the relationship between lengths of $\overline{BC}$ and $\overline{DC}$.

2. Write a conjecture based on your investigation in *Item 1*.

3. Use what you learned in the previous task about tangents to help you prove that your conjecture is always true. Once you have proven the conjecture, it becomes a theorem. Add this theorem and a diagram to your *Circle Book*.

Mathematics II
Investigating Lengths of Segments
Day 1 Homework

1. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE = 4\frac{2}{3}$, $CE = 3\frac{1}{7}$, and $ED = 2\frac{4}{9}$. Find EB.

2. A diameter of a circle is perpendicular to a chord whose length is 12 inches. If the length of the shorter segment of the diameter is 4 inches, what is the length of the longer segment of the diameter?

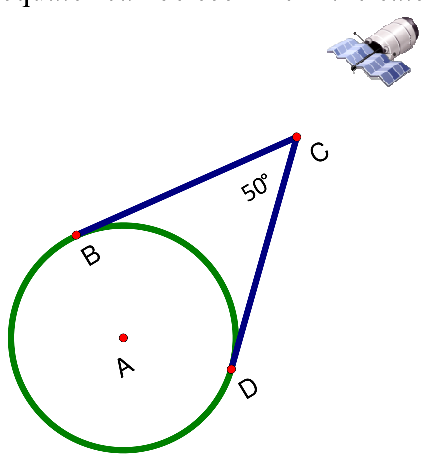
3. In a circle, $\overline{AB}$ bisects $\overline{CD}$ at point E. $\overline{CD}$ is 6 inches long. If $AE = x - 3$ and $EB = x + 5$, find AE, EB, and AB.

4. Two secant segments are drawn to a circle from a point outside the circle. The external segment of the first secant segment is 8 centimeters and its internal segment is 6 centimeters. If the entire length of the second secant segment is 28 centimeters, what is the length of its external segment?

5. A tangent segment and a secant segment are drawn to a circle from a point outside the circle. The length of the tangent segment is 15 inches. The external segment of the secant segment measures 5 inches. What is the measure of the internal secant segment?

6. The diameter of a circle is 19 inches. If the diameter is extended 5 inches beyond the circle to point *C*, how long is the tangent segment from point *C* to the circle?

7. A satellite orbits the earth so that it remains at the same point above the Earth's surface as the Earth turns. If the satellite has a 50° view of the equator, what percent of the equator can be seen from the satellite?



8. The average radius of the Earth is approximately 3959 miles.
   a. How far above the Earth's surface is the satellite described in *Problem 7?*
   b. What is the length of the longest line of sight from the satellite to the Earth's surface? Identify this line of sight using the diagram.

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Mathematics II: Unit 3
Circles and Spheres

Task 5: Finding Arc Length and the Area of a Sector

Mathematics II
Task 5: ***Finding Arc Length and the Area of a Sector*** **Day 1/1**

MM2G3. Students will understand the properties of circles.
c. Use the properties of circles to solve problems involving the length of an arc and the area of a sector.

New vocabulary: arc length, sector of a circle, concentric circles

Mathematical concepts/skills:
- finding the circumference of a circle
- finding the area of a circle
- finding the length of an arc as a portion of the circumference of a circle
- finding the area of a sector as a portion of the area of a circle

Prior knowledge:
- finding the circumference of a circle
- finding the area of a circle

Essential question(s): How can I find the length of an arc? How can I find the area of a sector? What kinds of real-world problems can be solved using these measures?

Suggested materials:
- unlined paper
- compass
- straightedge
- protractor

Warm-up: Post the following:

Circle P has a radius of 11 centimeters.

1. *Find the circumference of the circle. Give both the exact measure and an approximate measure rounded to the nearest hundredth.*

2. *Find the area of the circle. Give both the exact measure and an approximate measure rounded to the nearest hundredth.*

Opening: Discuss the opening, making sure that students remember the formulas for finding the circumference and area of a circle. Emphasize appropriate units for both measures.

Worktime: Students should work in pairs to complete *Problems 1 – 7*.

Allow students to answer the questions posed in Problems 1 and 2 by any means they choose. Hopefully, students will see that the two horses on the carousel are traveling different circles and will apply the concepts discussed in the warm-up. Ask guiding questions and allow ample time for students to develop their own understanding of arc length as a portion of the circumference of a circle.

After students have had time to derive a method or a formula for arc length, have a whole-class discussion of *Problems 1 -3* before moving on to the next set of questions. Some students may set up a proportion to find the length of an arc. Others may have developed a formula similar to the one given below. Make sure that both ideas are shared and discuss why the two methods provide the same results.

Arc length is determined as a portion of the total circumference of a circle:

$$L = \frac{a^0}{360^0} C$$

where 'L' represents the length of the arc, 'a' represents the number of degrees in the central angle intercepting the arc, and 'C' represents the circumference of the circle.

Before beginning problem 4, make sure that students understand the definition of a sector. As with arc length, allow students to answer the questions posed in *Problems 4 – 6* by any means they choose. Ask guiding questions and allow ample time for students to develop their own understanding of the area of a sector as a portion of the area of a circle.

Closing: After students have had time to derive a method or a formula for the area of a sector, have a whole-class discussion of *Problems 4 – 7*. Some students may set up a proportion to find the area. Others may have developed a formula similar to the one given below. Again make sure that both ideas are shared and discuss why the two methods provide the same results.

Area of a sector is determined as a portion of the total area of a circle:

$$A = \frac{a^0}{360^0} \pi r^2$$

where 'A' represents the area of the sector, 'a' represents the number of degrees in the central angle intercepting the arc, and 'r' represents the radius of the circle.

All students should include definitions and formulas in their *Circle Books*, along with diagrams.

Homework: See homework following the Student Task.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview;

- finding the circumference of a circle
- finding the area of a circle

Mathematics II
Finding Arc Length and the Area of a Sector
Day 1 Student Task

Arc Length
In the first task of this unit, we discussed the fact that arcs are measured in two different ways. The *measure* of an arc is calculated in units of degrees and is defined to be the measure of its central angle. Arc **length** is calculated in units of distance. In this task, you will develop a formula for calculating the length of an arc.

Consider the carousel in the picture above. The innermost horse in the picture is 12 feet from the center of the carousel. The outermost horse is 24 feet from the center.

1. Suppose the carousel makes one complete revolution.
   a. Through how many degrees does the outermost horse turn?

   b. Through how many degrees does the innermost horse turn?

   c. Do the two horses travel the same *distance*? Why or why not?

   d. If the two horses travel the same distance, how far do they travel? If they travel different distances, how far does each horse travel? Show how you know.

2. Suppose the carousel rotates through 120°.
   a. Through how many degrees does the outermost horse turn?

   b. Through how many degrees does the innermost horse turn?

   c. How far does each horse travel during this rotation? Show how you know.

3. The positions of the innermost and the outermost horses on the carousel can be modeled by two concentric circles. **Concentric** circles are coplanar circles with the same center.
   a. Use your compass to construct concentric circles that represent the positions of the innermost and outermost horses as the carousel rotates.
   b. Consider that the *distance* a horse travels is the *length* of the arc the horse traverses on its circle. Use your diagram and your answers to *Problems 1* and *2* to help you determine a formula for finding the length of any arc on any circle.

Area of a Sector

The carousel in the picture above needs refurbishing. Suppose, in an effort to make things colorful, the carnival owner wishes to paint a pattern of sectors on the carousel floor. A **sector** of a circle is a region between two radii and an arc of the circle.

4. Consider the floor of the carousel. It can be represented by the outer circle of your diagram in *Problem 3a*. Use your compass to construct a single circle that represents the floor of the carousel. What is the area of the floor? Show how you know?

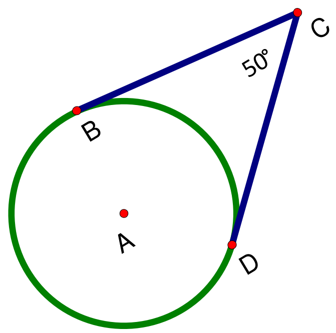
5. The owner has decided to paint the floor in a repeating pattern of sectors with central angles of 10°, 20°, and then 30°. Use you protractor and a straightedge to draw the pattern on your circle. How many sectors of each degree measure are on your "floor"?

6. Suppose each sector with a central angle of 10° will be painted purple, each sector with a central angle of 20° will be painted pink, and each sector with a central angle of 30° will be painted blue. How many square feet of the floor will be painted purple? pink? blue? Show how you know.

7. Use what you have learned in *Problems 4 – 6* to help you determine a formula for finding the area of any sector of any circle.

Mathematics II
Finding Arc Length and the Area of a Sector
Day 1 Homework

1. A satellite orbits the earth so that it remains at the same point above the Earth's surface as the Earth turns. The satellite has a 50° view of the equator. If the average radius of the Earth is 3959 miles, approximately how many *miles* of the equator can be seen from the satellite?

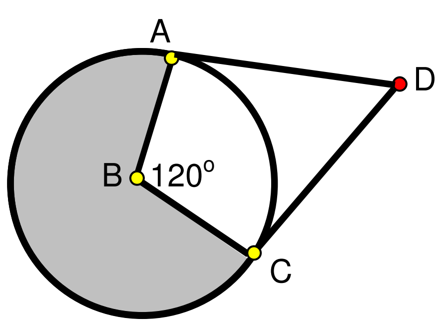


2. On the clock shown here, the hour hand is 4.5 inches long and the minute hand is 6 inches long.

a. What is the *measure* of the arc swept out by the hour hand as it moves from 11 on the face of the clock to 4? What is the *length* of this arc?

b. What is the *measure* of the arc swept out by the minute hand as it moves from 11 on the face of the clock to 4? What is the *length* of this arc?

c. What is the area of the sector swept out by the hour hand as it moves from 11 to 4?



3. In the diagram above, $\overline{AD}$ and $\overline{CD}$ are tangent to $\odot B$ at A and C respectively. $AD = 6\sqrt{3}$. Find the area of the shaded region.

4. Classify the following statement as always true, sometimes true, or never true. Justify your thinking.

   If two arcs have the same measure, then they are congruent.

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Mathematics II: Unit 3
Circles and Spheres

Task 6: Volume and Surface Area of a Sphere

Mathematics II
Task 6: ***Volume and Surface Area of a Sphere*** **Day 1/2**

MM2G4. Students will find and compare the measures of spheres.
a. Use and apply surface area and volume of a sphere.

New vocabulary: sphere, hemisphere, center of a sphere, radius of a sphere, great circle

Mathematical concepts/skills:

- definitions of sphere, hemisphere, center of a sphere, radius of a sphere, great circle
- deriving the formula for the volume of a sphere
- finding the volume of a sphere given the radius
- finding the radius of a sphere given the volume

Prior knowledge:

- definitions of center, radius, and diameter
- volume of a right circular cylinder

Essential question(s): How can I find the volume of a sphere? What kinds of real-world problems can be solved using the volume of a sphere?

Suggested materials:

- sphere and right circular cylinder, with diameter and height equal to the diameter of the sphere, that will hold water, sand, rice or other pourable substance and ruler for measuring height or
- Youtube video at the following website: http://www.youtube.com/watch?v=aLyQddyY8ik

Although we know many classrooms do not have the sphere and right circular cylinder mentioned above and the Youtube video is available as an alternative to the experiement, be aware that students will retain the information better if they actually experience the hands-on activity.

Warm-up: Post the following:

> *Sketch a right circular cylinder with radius 'r' and height '2r'. Find the volume of the cylinder. Explain how you found the volume.*

Opening: Discuss the warm-up, making sure that students not only remember the formula for finding the volume of a cylinder but also understand that the volume of any right prism is always the area of the base times the height.

Worktime: Students should complete *Problems 1 – 5* of the task along with the suggested investigation.

Allow students to work in pairs to read and complete *Problems 1 – 3* of the task. A whole-class discussion of these problems should be held before students begin their investigation or watch the suggested video.

After students demonstrate an understanding of the terms introduced at the beginning of the task, they should begin the investigation. Make sure that all students have derived the correct formula for the volume of a sphere before beginning *Problems 4 and 5*.

Closing: Have a thorough discussion of the investigation performed in this task, the formula derived for the volume of a sphere, and the applications of the formula provided in *Problems 4* and *5*.

Homework: Students should be assigned homework problems related to the understanding and real-world applications of the volume of a sphere.

Differentiated support/enrichment:

Check for Understanding:

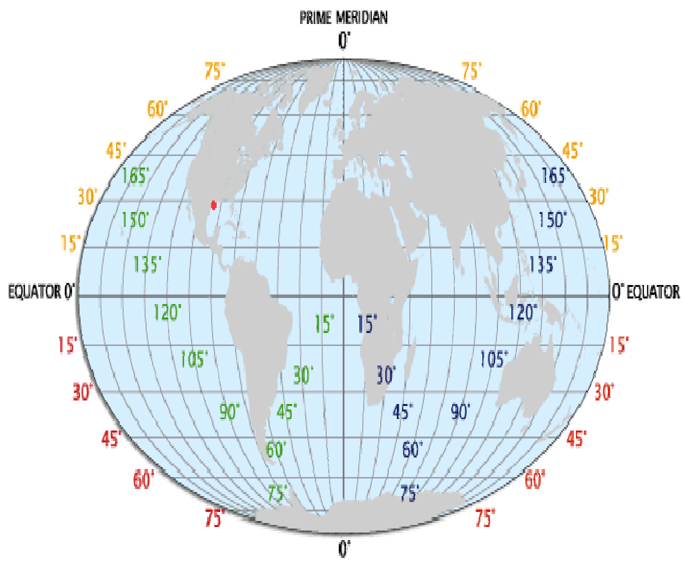
Resources/materials for Math Support: Students should preview finding the volume of a right circular cylinder.

Mathematics II
Volume and Surface Area of a Sphere
Day 1 Student Task

In this task we will examine the volume and surface area of a sphere but first we need to define some terms that will be used throughout the lesson.

A **sphere** is a solid with a curved surface. Examples include a tennis ball, a marble, a basketball and a globe. Remember that we defined a circle as the set of all points in a *plane* equidistant from a given point (the center). We can define a sphere in a similar manner. A sphere is the set of all points in *space* equidistant from a given point. The given point is called the **center** and the distance is the **radius.**

Any circle on a sphere that has the same center and radius as the sphere is called a **great circle** of the sphere. A **hemisphere** is half of a sphere and the great circle that forms its base.



Consider the Earth as a perfect sphere, which we often do even though this is not exactly the case.

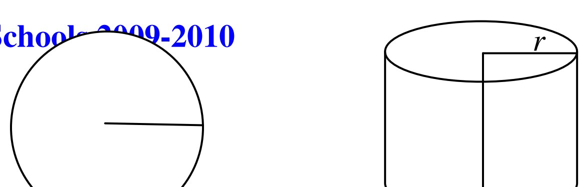
1. What is meant by “the Northern Hemisphere?

2. On the picture of the Earth above, the Prime Meridian is shown. *Longitudinal* lines on the globe run north and south, “parallel” to the Prime meridian. Lines of *latitude* run east and west, “parallel” to the Equator. Use the picture above to help you discuss which “lines” form great circles of the Earth and which do not.

3. The **Prime Meridian** is the line at which longitude is defined to be 0°. Other lines of longitude are measured east or west of the Meridian and are usually expressed in degrees (or hours), minutes, and seconds. If the radius of the Earth is approximately 3959 miles, how many miles of the Equator lie between the Prime meridian and the longitudinal line that is 45° east of the Meridian?

Volume of a Sphere

We will derive the formula for the volume of a sphere in the same manner it was first derived by Archimedes around 250 B.C. - by comparing the capacity of a sphere and a right circular cylinder with radius equal to that of the sphere and height twice the radius of the sphere. In the



pictures shown here, we label the radius of the sphere and the cylinder as *r*. The height of the cylinder is 2*r*.

r

Use the materials provided by your teacher to perform the following investigation:

1. Fill the sphere and then carefully pour the contents of the sphere into the cylinder.
2. Determine the part of the cylinder filled by the contents of the sphere.
3. Use the formula for the volume of the cylinder and what you learned in *Steps 1* and *2* to derive a formula for the volume of a sphere.

Use the formula you derived for the volume of a sphere to solve the following problems:

4. Find the volume of a sphere with radius of 5 inches. Give both the exact value and an approximation to the nearest tenth of a unit.

5. The volume of a sphere is given to be approximately 546 in^3. Find the radius of the sphere to the nearest tenth of an inch.

Mathematics II

Task 6: *Volume and Surface Area of a Sphere* **Day 2/2**

MM2G4. Students will find and compare the measures of spheres.

a. Use and apply surface area and volume of a sphere.

b. Determine the effect on surface area and volume of changing the radius or diameter of a sphere.

New vocabulary:

Mathematical concepts/skills:
- deriving the formula for the surface area of a sphere
- finding the surface area of a sphere given the radius
- finding the radius of a sphere given the surface area

Prior knowledge:
- definitions of center, radius, and diameter
- surface area of a right circular cylinder

Essential question(s): How can I find the surface area of a sphere? What kinds of real-world problems can be solved using the surface area of a sphere?

Suggested materials:
- oranges
- construction paper
- string
- scissors
- rulers
- wax paper, plastic wrap, or garbage bags (To place under student work areas.)
- pail/bowl of soapy water and paper towels (For students to rinse hands.)

Warm-up: Post the following:

> *Sketch a right circular cylinder with radius 'r' and height '2r'. Find the surface area of the cylinder. Explain how you found the surface area.*

Opening: Discuss the warm-up, making sure that students not only remember the formula for finding the surface area of a cylinder but also understand that the surface area is comprised of the area of a rectangle that makes up the lateral surface of the cylinder and the areas of the two congruent circles that form the bases of the cylinder.

Worktime: Students should perform the investigation and complete *Problems 6 – 13* of the task.

A whole-class discussion of the investigation and the formula derived should be held before students begin *Problem 6* of the task. Likewise, a discussion should be held after *Problem 7* to be sure students can apply the formula before moving on to explore the effects of a change in the radius on surface area and volume.

Closing: In discussing *Problems 8 – 13*, it is important students understand that multiplying the radius of a sphere by *n,* multiplies the surface area by $n^2$and the volume by n^3. Students are led to see this numerically first and then asked to generalize their findings.

Homework: Students should be assigned homework problems related to the understanding and real-world application of the surface area of a sphere.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview finding the surface area of a right circular cylinder.

Skills problems related to finding the volume and surface area of a sphere can be found at http://www.geogebra.org/en/upload/files/english/lewisprisco/sphere.html .

Mathematics II
Volume and Surface Area of a Sphere
Day 2 Student Task

Surface Area of a Sphere

To explore the formula for the surface area of a sphere, we will perform a second experiment.

i. As closely as possible, find the measure of the diameter of the orange given to you by your teacher. (You might want to discuss the best way of doing this before you begin.)

ii. From a piece of construction paper, cut a rectangle that would form the *lateral surface* of a cylinder similar to the one used in the previous investigation. The diameter and the height of the cylinder should be equal to the diameter of your orange. Once it is cut, wrap the rectangular paper around the orange to verify diameter and height.

iii. Peel the orange. As you peel, fit the pieces of the peeling on the rectangle covering the paper in one layer as closely as possible. What happens? Compare results with several other groups of students.

iv. Use what you have discovered in *Steps i – iii* of this experiment to help you write a formula for the surface area of a sphere with radius *r*.

Use the formula you derived for the surface area of a sphere to solve the following problems:

6. Find the surface area of a sphere with a radius of 5.5 centimeters.

7. The surface area of a sphere is approximately 84 in^2. Find the radius of the sphere.

The Effect of Changing the Radius of a Sphere

Now that you know how to find the volume and surface area of a sphere, we will investigate one more important question.

How does a change in the radius of a sphere affect its surface area and volume?

Let's first consider the surface area.

8. Suppose the radius of a sphere is multiplied by 3. How does this change the surface area of the sphere?

9. Suppose the radius of a sphere is multiplied by four-fifths. How does this change the surface area?

10. Generalize your results. Suppose the radius of a sphere in multiplied by *n*. How does this change the surface area of the sphere? Show how you know.

Consider the same questions for the volume of a sphere.

11. Suppose the radius of a sphere is multiplied by 3. How does this change the volume of the sphere?

12. Suppose the radius of a sphere is multiplied by four-fifths. How does this change the volume?

13. Generalize your results. Suppose the radius of a sphere in multiplied by *n*. How does this change the volume of the sphere? Show how you know.